DOI: https://doi.org/10.46354/i3m.2019.emss.013

ON ESTIMATION ERRORS WHEN DEALING WITH THE PROBLEMS OF OPTICAL TELECOMMUNICATIONS

O.V. Chernoyarov[(a)], Yu.A. Kutoyants[(b)], A.V. Zyulkov[(c)]

[(a)]National Research University “Moscow Power Engineering Institute”, Moscow, Russia
[(a),(b)]International Laboratory of Statistics of Stochastic Processes and Quantitative Finance of the National Research Tomsk State University, Tomsk, Russia
[(a)]Maikop State Technological University, Maikop, Russia
[(b)]Le Mans University, Le Mans, France
[(c)]Voronezh State University, Voronezh, Russia

[(a)]chernoyarovov@mpei.ru, [(b)]kutoyants@univ-lemans.fr, [(c)]zyulkov@phys.vsu.ru

ABSTRACT
The problem is considered of the phase and frequency estimation by the observations of periodic Poisson processes in the cases of different regularity conditions: smooth signals, cusp-type singular signals and change-point type signals. There is described the asymptotic behavior of the mean square errors in all these situations and then the results of numerical simulations are presented.

Keywords: Poisson signal, maximum likelihood estimator, Bayesian estimator, phase modulation, frequency modulation, regular, cusp-type and discontinuous cases, rate of convergence of estimator

1. PROBLEM STATEMENT

1.1. The model of the realization of the observable data

Let us consider the problem of transmission of information ϑ_0 by a Poisson channel with the dark noise having the intensity of $\lambda_N > 0$. It is presupposed that the optical signal (inhomogeneous Poisson process) has an intensity function $S(\vartheta_0, t)$, $0 \le t \le T$, where $S(\vartheta_0, t)$ is periodic function. Therefore, the observations $X^T = (X_t, 0 \le t \le T)$ is Poisson process with the intensity function

$$\lambda(\vartheta_0, t) = S(\vartheta_0, t) + \lambda_N, \quad 0 \le t \le T.$$

The behavior of the mean square error (variance) is studied under $T \to \infty$:

$$\mathbf{E}_\vartheta \left(\bar{\vartheta}_T - \vartheta_0\right)^2 = C / T^\gamma, \qquad (1)$$

where $\bar{\vartheta}_T$ is a certain estimator of the parameter ϑ_0, $C > 0$ is a constant and the value $\gamma > 0$ depends on the regularity of the function $S(\vartheta_0, t)$ with respect to ϑ_0.

1.2. The estimators of the informative parameter

Let us introduce the likelihood ratio function

$$L\left(\vartheta, X^T\right) = \exp\left[\int_0^T \ln\left(1 + \frac{S(\vartheta, t)}{\lambda_N}\right) \mathrm{d}X_t - \int_0^T S(\vartheta, t)\,\mathrm{d}t\right].$$

Then, the two most commonly used estimators of the parameter ϑ_0 can be formed: maximum likelihood estimator (MLE) $\hat{\vartheta}_T$:

$$L\left(\hat{\vartheta}_T, X^T\right) = \sup_{\vartheta \in \Theta} L\left(\vartheta, X^T\right) \qquad (2)$$

and Bayesian estimator (BE)

$$\tilde{\vartheta}_T = \int_\Theta \vartheta p(\vartheta) L\left(\vartheta, X^T\right) \mathrm{d}\vartheta \Big/ \int_\Theta p(\vartheta) L\left(\vartheta, X^T\right) \mathrm{d}\vartheta.$$

Here Θ is the area of possible values, $p(\vartheta)$ is the positive continuous prior probability density of the parameter ϑ_0 and the quadratic loss is presupposed.

1.3. Modulations

P. *Phase modulation*: $S(\vartheta, t) = f(t - \vartheta)$
F. *Frequency modulation*: $S(\vartheta, t) = f(\vartheta t)$
Here and hereinafter $f(t)$ is the periodic function of the known period.

1.4. Regularity

S. *Smooth case*. The function $f(t)$ is continuously differentiable.
C. *Cusp-type case*. The function $f(t)$ has the following representation on the first period: $f(t) = a|t|^\kappa + h(t)$,

Proceedings of the European Modeling and Simulation Symposium, 2019
ISBN 978-88-85741-25-6; Affenzeller, Bruzzone, Longo and Pereira Eds.

where $a > 0$, $\kappa \in (0, 1/2)$, and $h(t) > 0$ is continuously differentiable.

D. *Discontinuous case*. The function $f(t)$ has the following representation on the first period: $f(t) = a\,1_{\{t > \vartheta\}}$.

The following theoretical results have been obtained for the MLE $\hat{\vartheta}_T$ and BE $\tilde{\vartheta}_T$:

PS case, i.e. phase modulation and smooth $f(t)$ (Kutoyants 1979):

$$\mathbf{E}_\vartheta \left(\hat{\vartheta}_T - \vartheta_0\right)^2 \sim C/T, \quad \gamma = 1. \tag{3}$$

PC case, i.e. phase modulation and cusp-type $f(t)$ (Dachian 2003):

$$\mathbf{E}_\vartheta \left(\hat{\vartheta}_T - \vartheta_0\right)^2 \sim C/T^{2/(2\kappa+1)}, \; 1 < \gamma = 2/(2\kappa + 1) < 2. \tag{4}$$

PD case, i.e. phase modulation and discontinuous $f(t)$ (Kutoyants 1979):

$$\mathbf{E}_\vartheta \left(\hat{\vartheta}_T - \vartheta_0\right)^2 \sim C/T^2, \quad \gamma = 2. \tag{5}$$

FS case, i.e. frequency modulation and smooth $f(t)$ (Kutoyants 1979):

$$\mathbf{E}_\vartheta \left(\hat{\vartheta}_T - \vartheta_0\right)^2 \sim C/T^3, \quad \gamma = 3. \tag{6}$$

FC case, i.e. frequency modulation and cusp-type $f(t)$:

$$\mathbf{E}_\vartheta \left(\hat{\vartheta}_T - \vartheta_0\right)^2 \sim \frac{C}{T^{(4\kappa+4)/(2\kappa+1)}}, \; 3 < \gamma = \frac{4\kappa + 4}{2\kappa + 1} < 4. \tag{7}$$

FD case, i.e. frequency modulation and discontinuous $f(t)$ (Kutoyants 1984):

$$\mathbf{E}_\vartheta \left(\hat{\vartheta}_T - \vartheta_0\right)^2 \sim C/T^4, \quad \gamma = 4. \tag{8}$$

With the use of simulations, the convergences

$$\ln \mathbf{E}_\vartheta \left(\hat{\vartheta}_T - \vartheta_0\right)^2 / \ln T \to -\gamma$$

are studied in all the cases specified above. The intensity functions in simulations are always

$$\lambda(\vartheta_0, t) = S(\vartheta_0, t) + 1, \quad 0 \le t \le n \to \infty, \tag{9}$$

where $S(\vartheta, t) = f(t - \vartheta)$ and $S(\vartheta, t) = f(\vartheta t)$ in the cases of phase and frequency modulations, respectively.

The function $f(t)$ in *smooth* **PS** and **FS** cases of the phase and frequency modulations is $f(t) = a\cos^2(2\pi t)$, $a = 2$, $t \ge 0$.

The function $f(t)$ in *cusp* **PC** and **FC** cases is periodic of period 1 and it has the representation on the one period $t \in [-1/2, 1/2]$ as follows

$$f(t) = \frac{a}{2}\left[1 + \operatorname{sgn}(2t + \delta)\left|\frac{2t + \delta}{\delta}\right|^\kappa 1_{\{-\delta < t \le 0\}} - \operatorname{sgn}(2t - \delta)\left|\frac{2t - \delta}{\delta}\right|^\kappa 1_{\{0 < t \le \delta\}}\right] 1_{\{-\delta < t \le \delta\}}. \tag{10}$$

Here $a = 2$, $\delta = 0.25$ and $\kappa = 1/4$.

The function $f(t)$ in *change-point* **PD** and **FD** cases of the phase and frequency modulations is periodic with the period 1:

$$f(t) = a\,1_{\{0 \le t \le \delta\}},$$

where $a = 10$, $\delta = 0.25$ and it is periodically prolonged along the whole line.

In Fig. 1, the qualitative examples are shown of the signals with varying degree of smoothness (different types of singularity of intensity functions (10)) determined by the parameter κ: a) $\kappa = 5/8$; b) $\kappa = 1/2$; c) $\kappa = 1/8$; d) $\kappa = 0$; e) $\kappa = -3/8$.

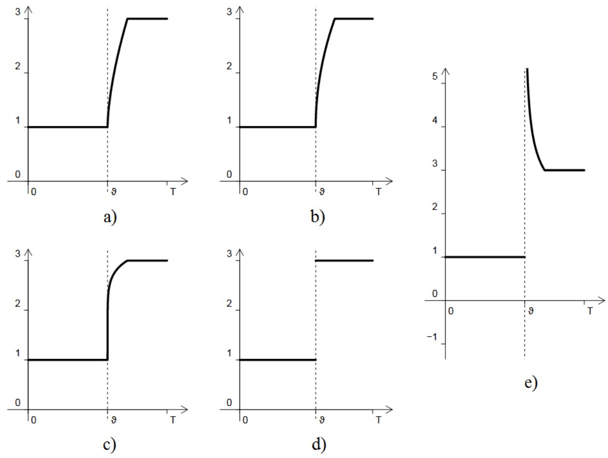

Figure 1: The examples of the signals with different types of singularity of intensity functions

The curve presented in Fig. 1a corresponds to the smooth case. Note that the derivative of this function is unbounded, but nevertheless the Fisher information is finite and this is a regular statistical experiment. The curve presented in Fig. 1b is like the smooth one as well, but the rate of convergence of the mean square error is slightly different: $\mathbf{E}_\vartheta \left(\hat{\vartheta}_T - \vartheta_0\right)^2 \sim C/T \ln T$. The curves presented in Figs. 1c and 1d correspond to cusp-type and change-point type singularities, respectively. Finally, the curve presented in Fig. 1e corresponds to the explosion-type singularity, the properties of the estimators are also known in this case, but this type of singularity is not examined in our present research.

ISBN 978-88-85741-25-6; Affenzeller, Bruzzone, Longo and Pereira Eds.

2. RESULTS OF SIMULATIONS

The convergence of the experimental values of the variances of MLE (2) with the asymptotic formulas (3)-(8) have been studied using simulation in programming language R. For non-stationary Poisson process, the simulation is carried out by the method of time scale conversion (Law 2014).
In Figs. 2-6, the log-log plots (1)

$$V = V(\gamma, T) = \ln \mathbf{E}_\vartheta \left(\hat{\vartheta}_T - \vartheta_0\right)^2 \sim \ln C - \gamma \ln T, \qquad (11)$$

of the variance of MLE (2) as a function of the number of signal periods N are presented where one can see some results of statistical simulation (circles, squares, pluses) and corresponding theoretical dependences (solid curves). The C parameter is estimated based on the simulation data by the least squares method. Each experimental value is obtained by processing no less than 10^3 realizations of Poisson process with the intensity function (9). Thus, with the probability of 0.9, the confidence intervals boundaries deviate from the experimental values no greater than by 10 %.
In Fig. 2, the dependences (11) are presented for all the three types of regularity in case of the phase modulations (**P**): smooth (3) – curve 1 and circles, cusp-type (4) – curve 2 (if $\gamma = 4/3$) and squares, change-point (5) – curve 3 and pluses.
Fig. 3 describes the similar situations but for the intensities with the frequency modulation (**F**): smooth (6) – curve 1 and circles, cusp-type (7) – curve 2 (if $\gamma = 10/3$) and squares, change-point (8) – curve 3 and pluses.
In the next three figures, there is shown the comparison of the errors (11) occurring in the cases when the same regularity is accompanied with different types of modulations.
In Fig 4, the theoretical and experimental errors (11) are plotted for the cases of the phase (curve 1 and pluses) and frequency (curve 2 and pluses) modulations for smooth intensity functions, i.e. for **PS** and **FS** cases described by the formulas (3) and (6), respectively.

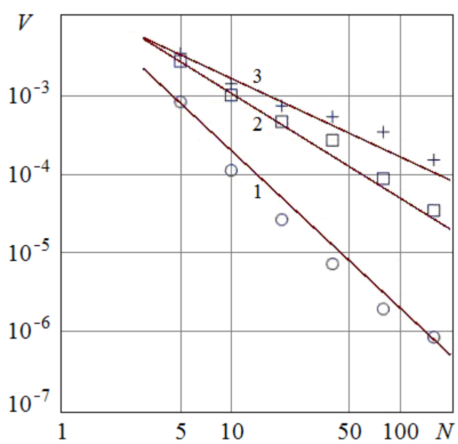

Figure 2: The dependences of the variance of MLE of the Poisson signal in the case of phase modulations

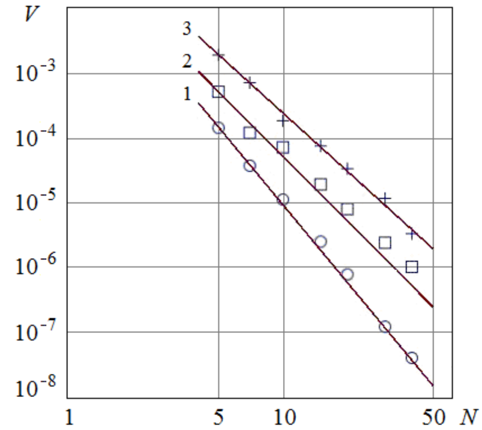

Figure 3: The dependences of the variance of MLE of the Poisson signal in the case of frequency modulations

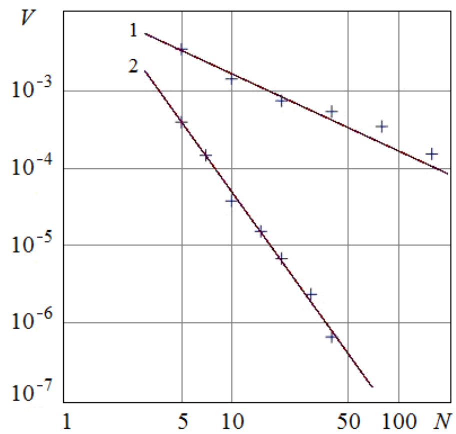

Figure 4: The dependences of the variance of MLE of the Poisson signal with smooth intensity function in the case of phase and frequency modulations

The results presented in Fig. 5 correspond to the phase (curve 1 and squares) and frequency (curve 2 and squares) modulations for the intensities with cusp-type singularity, i.e. for **PC** and **FC** cases described by the formula (3) under $\gamma = 4/3$ and the formula (6) under $\gamma = 10/3$, respectively.

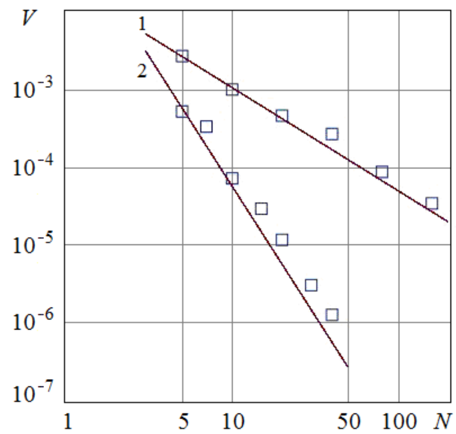

Figure 5: The dependences of the variance of MLE of the Poisson signal with cusp-type intensity function in the case of phase and frequency modulations

And finally, in Fig. 6, the dependences (11) are drawn for the phase-modulated (curve 1 and circles) and frequency-modulated (curve 2 and circles) signals with discontinuous intensities, i.e. for **PD** and **FD** cases described by the formulas (5) and (8), respectively.

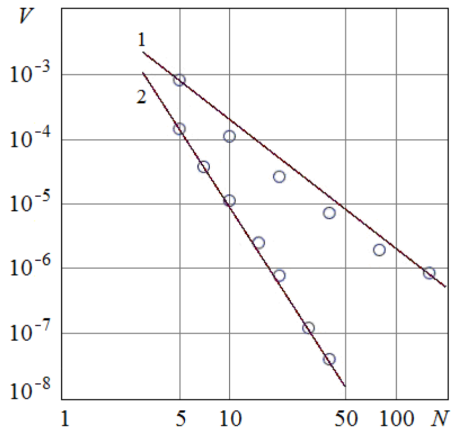

Figure 6: The dependences of the variance of MLE of the Poisson signal with discontinuous intensity function in the case of phase and frequency modulations

One can see that the result of the simulations correspond well to the theoretical properties (3)-(8) of the mean square errors in a wide range of observations' duration.

3. CHOICE OF THE MODEL

Therefore, it is natural to put the following question: *What is the best choice of the intensity function and the estimator and what is the corresponding rate of decay of the mean square error*?

It should be noted that in statistics the observation model is usually presented and the problem is to understand what can be done to identify this model. Here the statement is different and one can choose the model to get less errors.

The problem considered there is, in some sense, inverse. It is presupposed that one can choose any intensity one wants, and the goal is to find such function $\lambda(\vartheta_0,t)$ of $\vartheta_0 \in \Theta = (0,1)$ and $t \in [0,T]$ and the estimator ϑ_T^* that the rate of error decreasing is the best possible. Of course, one has to impose some restrictions on the "energy of the signal" (in terminology coming from telecommunication theory), since, if one allows that $\lambda(\vartheta_0,t) \to \infty$, then one will have any rate wanted.

Let us fix some number $L > 0$ and introduce the class of intensity functions bounded by this constant

$$F(L) = \{\lambda(\cdot):\ 0 \le \lambda(\vartheta,t) \le L,\ 0 \le t \le T\}.$$

Then one gets the following result:

$$\inf_{\lambda \in F(L)} \inf_{\bar{\vartheta}_T} \sup_{\vartheta_0 \in \Theta} \mathbf{E}_{\lambda,\vartheta_0} \left(\bar{\vartheta}_T - \vartheta_0\right)^2 = \exp\left[-\frac{TL}{6}(1+o(1))\right].$$

This relation contains two different results.

The first one is a lower bound on the risks for all the choices of the intensity function (in $F(L)$) and all the estimators $\bar{\vartheta}_T$:

$$\inf_{\lambda \in F(L)} \sup_{\vartheta_0 \in \Theta} \mathbf{E}_{\lambda,\vartheta_0} \left(\bar{\vartheta}_T - \vartheta_0\right)^2 \ge \exp\left[-\frac{TL}{6}(1+o(1))\right].$$

The second result is to describe the intensity function $\lambda_*(\cdot)$ and such the estimator ϑ_T^* that (upper bound)

$$\sup_{\vartheta_0 \in \Theta} \mathbf{E}_{\lambda_*,\vartheta_0} \left(\vartheta_T^* - \vartheta_0\right)^2 \le \exp\left[-\frac{TL}{6}(1+o(1))\right].$$

Proofs can be found in the researches by Burnashev and Kutoyants (1999, 2001). Note that this result follows *the spirit of Information Theory*.

4. CONCLUSION

The goal of this research is to show how the rate of decreasing the mean square error of MLE depends on the regularity of the model. The large diversity of the rates can be seen due to the different types of the smoothness of the signals. There is another important class of problems: what are the errors of the estimators when the regularities of the statistical models real and theoretical are different. It is also proposed to present some results in such situations.

ACKNOWLEDGMENTS

This research was financially supported by the Ministry of Education and Science of the Russian Federation (research project No. 2.3208.2017/4.6).

REFERENCES

Kutoyants Yu.A., 1979. Intensity parameter estimation of an inhomogeneous Poisson process. Problems of Control and Information Theory, 8 (2), 1-13.

Dachian S., 2003. Estimation of cusp location by Poisson observations. Statistical Inference for Stochastic Processes, 6 (1), 1-14.

Kutoyants Yu.A., 1984. Parameter estimation for stochastic processes. Berlin: Heldermann.

Law A.M., 2014. Simulation modeling and analysis. New York: McGrow-Hill Education.

Burnashev M.V. and Kutoyants Yu.A., 1999. On sphere-packing bound, capacity and related results for Poisson channel. Problems of Information Transmission, 35 (2), 3-22.

Burnashev M.V. and Kutoyants Yu.A., 2001. On minimal α-mean error parameter transmission over Poisson channel. IEEE Transactions on Information Theory, 47 (6), 2505-2515.

AUTHORS BIOGRAPHY

Oleg V. Chernoyarov was born in 1973. He received his M.Sc. and Ph.D. degrees in radio physics from the

Voronezh State University, Voronezh, Russia, and the D.Sc. degree in radio physics – from the Moscow Pedagogical State University, Moscow, Russia, in 1996, 2000 and 2011, respectively. Presently, he is working as a Professor at the Department of Electronics and Nanoelectronics of the National Research University "Moscow Power Engineering Institute" as well as a Leading Research Worker of the National Research Tomsk State University (International Laboratory of Statistics of Stochastic Processes and Quantitative Finance) and the Maikop State Technological University (Department of Higher Mathematics and System Analysis). His main interests are statistical methods for transmission and processing of information.

Yury A. Kutoyants was born in 1946. He received his M.Sc. degree in nuclear physics from Moscow Engineering and Physics Institute (MIFI), he defended his Ph.D. degree in information theory from Moscow Institute of Physics and Technology and the D.Sc. degree in mathematics from Leningrad State University in 1970, 1975 and 1988 respectively. Presently he is working as Emeritus Professor at the Le Mans University, France as well as a Leading Research Worker of the National Research Tomsk State University (International Laboratory of Statistics of Stochastic Processes and Quantitative Finance). His main interests are in statistics of continuous time stochastic processes.

Aleksandr V. Zyulkov was born in 1952. He received his M.Sc. and Ph.D. degrees in radio physics from the Voronezh State University, Voronezh, Russia, in 1974 and 1983, respectively. Presently, he is working as an Associate Professor at the Department of Radio Physics of the Voronezh State University. His main interests are statistical methods for transmission and processing of information.