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DEFENDER-ATTACKER-TARGET GAME: FIRST-ORDER DEFENDER AND ATTACKER DYNAMICS

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ABSTRACT
Based on the solution of a linear-quadratic differential game with a terminal attacker's constraint, obtained in the previous paper, the practically important case of first-order players' dynamics is treated. The game space decomposition is constructed. The fulfillment of the saddle point inequalities is demonstrated. The feedback realization of the optimal strategies is presented.

Keywords: pursuit-evasion differential game, zero-sum linear-quadratic game, terminal constraint, first-order dynamics

1. INTRODUCTION

A defender-attacker-target problem is a widely discussed topic in the control and guidance literature (see e.g., (Rubinsky and Gutman 2014; Garcia, Casbeer, and Pachter 2017) and others). In the previous paper of the authors (Turetsky and Glizer 2019), one can find a detailed literature review on different approaches for the modeling and solution of this problem.

In (Turetsky and Glizer 2019), a linear-quadratic differential game with an attacker's terminal constraint was considered. It was assumed that the player's controllers are described by linear differential equations of an arbitrary order. In this game, the objective of the defender (pursuer) is to capture the attacker (evader), i.e. to nullify the miss distance (the closest separation between the vehicles), while the evader tries to avoid the capture. However, in a practical situation, avoiding the capture is not the main aim of the evader. Its actual aim is to hit a prescribed static object (target), while avoiding the capture is an auxiliary aim (see, e.g., (Lipman and Shinar 1995) and references therein). Since the evader tries not only to escape the pursuer, but also to hit the target, it should be able to reach the target after the interception moment. This is described by a terminal state inequality constraint.

The general solution of the corresponding linear-quadratic differential game with the terminal evader's constraint was obtained in the previous paper of the authors (Turetsky and Glizer 2019). It was shown that subject to a condition on the evader's penalty coefficient in the cost functional, the game space is decomposed into three non-intersecting regions of different saddle point solutions.

In the literature, different types of missiles' dynamics can be found. The zero-order evader's dynamics ("ideal" evader) is traditionally interpreted as a worst case for the pursuer. The special case, where the controller dynamics of both the pursuer and the evader is zero-order, was considered in (Rubinsky and Gutman 2014; Glizer and Turetsky 2015). In (Lipman and Shinar 1995), the evader is ideal, whereas the pursuer has the first-order dynamics. However, real life controllers cannot transfer the control command into the missile acceleration instantaneously. Therefore, the zero-order dynamics model has rather theoretic implementation. In this paper, the special case where both players have first-order dynamics is elaborated. The first-order dynamics of the players models an intrinsic controller property: a time lag between the control command and the lateral acceleration. Thus, this case represents a realistic model of missiles engagement as emphasized by Shinar (1981), which makes it very important from the practical point of view. The first-order approximation of the pursuer's and the evader's dynamics was exploited in numerous papers on vehicles guidance and control (see, e.g., (Shinar 1981; Shinar, Glizer, and Turetsky 2013) and references therein).

2. PREVIOUS RESULTS

In this section, the results of Turetsky and Glizer (2019) on the solution of the game with arbitrary order of the players' controllers are briefly outlined.

2.1. Original Pursuit-Evasion Game

The engagement between the defender (pursuer) and the attacker (evader) is considered. In Fig. 1, the schematic engagement geometry is depicted. The X - axis is the initial line of sight. The Y -axis is normal to the X - axis. The origin of the coordinate system is collocated with the target position, which is also the initial position of the pursuer. The points (x_p, y_p) and (x_e, y_e) are current coordinates of the pursuer and the evader, respectively; V_p, V_e are their velocities; a_p, a_e are their lateral accelerations; φ_p, φ_e are the respective angles between the velocity vectors and the X -axis.

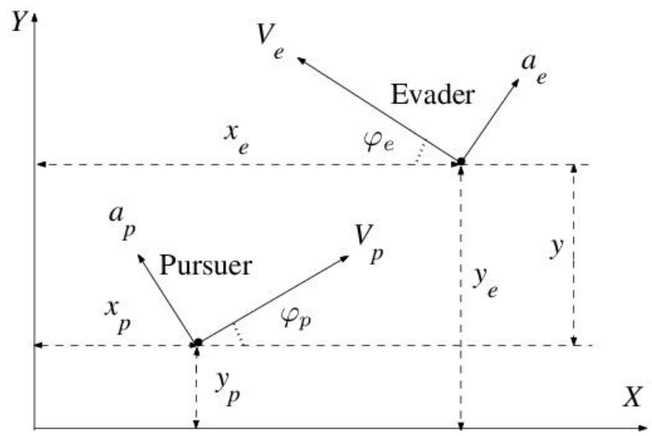

Figure 1: Interception Geometry

The controller dynamics of the pursuer and the evader are described by the equations

$$\dot{\bar{x}}_i = \bar{A}_i \bar{x}_i + \bar{b}_i u_i, \ \bar{x}_i(0) = [0]_{n_i \times 1}, \ i = p, e, \tag{1}$$

$$a_i = \bar{c}_i^T \bar{x}_i + \bar{d}_i u_i, \ i = p, e, \tag{2}$$

where $\bar{x}_i$ is the state vector consisting of n_i internal variables, u_i is the scalar control, $i = p, e$; a_p and a_e are the lateral accelerations of the pursuer and the evader, respectively, $[0]_{k\times m}$ denotes a zero $(k \times m)$-matrix. In the equations (1) – (2), $\bar{A}_i$ is a given constant matrix, $\bar{b}_i$ and $\bar{c}_i$ are given constant vectors, $\bar{d}_i$ is a given scalar.

The system dynamics for $t \in [0, t_f]$ is described by the linear differential equations of motion of the pursuer and the evader:

$$\dot{X}_i = A_i X_i + B_i u_i, \ i = p, e \tag{3}$$

where the state vector is $X_i = [y_i, \dot{y}_i, \bar{x}_i^T]^T$,

$$A_i = \begin{bmatrix} 0 & 1 & [0]_{1\times n_i} \\ 0 & 0 & \bar{c}_i^T \\ [0]_{n_i \times 1} & [0]_{n_i \times 1} & \bar{A}_i \end{bmatrix}, \tag{4}$$

$$B_i = \begin{bmatrix} 0 \\ \bar{d}_i \\ \bar{b}_i \end{bmatrix}, i = p, e. \tag{5}$$

The initial condition is

$$X_i(0) = [0, V_i \varphi_i(0), [0]_{1\times n_i}]^T, \ i = p, e. \tag{6}$$

The objective of the pursuer is to minimize the cost functional

$$J = (y_e(t_f) - y_p(t_f))^2 + \alpha \int_0^{t_f} u_p^2(t)dt - \beta \int_0^{t_f} u_e^2(t)dt, \tag{7}$$

where $\alpha, \beta > 0$ are penalties for the players' controls. The evader's first objective is to maximize (7). The second objective is to be capable to reach the target at $t = t_f + t_c$, i.e. to satisfy the constraint

$$| D_e \Phi_e (t_f + t_c, t_f) X_e(t_f) | \leq \mu_e a_e^{\max}, \tag{8}$$

where $t_c = \nu t_f$, $\nu = V_p / V_e$,

$$D_e = \left[1, [0]_{1\times(n_e+1)} \right], \tag{9}$$

$$\mu_e = \int_{t_f}^{t_f+t_c} \left| D_e \Phi_e (t_f + t_c, t) B_e \right| dt, \tag{10}$$

$\Phi_e(t, \tau)$ is the transition matrix of the homogeneous system, corresponding to (3) for $i = e$. For $t \in [t_f, t_f + t_c]$, it is assumed that

$$| u_e(t) | \leq a_e^{\max}. \tag{11}$$

The pursuit-evasion differential game for the system (3) with the cost functional (7) and the evader's terminal constraint (8) is called the Original Pursuit-Evasion Game (OPEG).

2.2. Reduced Game

The relative motion between the evader and the pursuer in the direction normal to the initial line-of-sight (the Y axis direction) is described by the system

$$\dot{X}_{ep} = A_{ep} X_{ep} + B_{ep} u_p + C_{ep} u_e, \tag{12}$$

where $X_{ep} = [y_e - y_p, \dot{y}_e - \dot{y}_p, x_p^T, x_e^T]^T$,

$$A_{ep} = \begin{bmatrix} 0 & 1 & [0]_{1\times n_p} & [0]_{1\times n_e} \\ 0 & 0 & -c_p^T & c_e^T \\ [0]_{n_p \times 1} & [0]_{n_p \times 1} & A_p & [0]_{n_p \times n_e} \\ [0]_{n_e \times 1} & [0]_{n_e \times 1} & [0]_{n_e \times n_p} & A_e \end{bmatrix},$$

$$B_{ep} = \begin{bmatrix} 0 \\ -\bar{d}_p \\ \bar{b}_p \\ [0]_{n_e \times 1} \end{bmatrix}, C_{ep} = \begin{bmatrix} 0 \\ \bar{d}_e \\ [0]_{n_p \times 1} \\ \bar{b}_e \end{bmatrix}.$$

Let $D_{ep} = \left[1, [0]_{1\times(n_p + n_e + 1)} \right]$, $\Phi_{ep}(t_f, t)$ be the transition matrix of the homogeneous system, corresponding to (12). New scalar state variables

$$z(t) = D_{ep} \Phi_{ep} (t_f, t) X_{ep}(t), \tag{13}$$

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$$w(t) = D_e \Phi_e (t_f + t_c, t) X_e(t), \quad (14)$$

satisfy the differential equations

$$\dot{z} = h_p(t)u_p + h_e(t)u_e, \; z(0) = z_0, \quad (15)$$

$$\dot{w} = g_e(t)u_e, \; w(0) = w_0, \quad (16)$$

where

$$h_p(t) = D_{ep}\Phi_{ep}(t_f, t)B_{ep}, \quad h_e(t) = D_{ep}\Phi_{ep}(t_f, t)C_{ep}, \quad (17)$$

$$g_e(t) = D_e \Phi_e(t_f + t_c, t)B_e, \quad (18)$$

$$z_0 = t_f(V_e\varphi_e^0 - V_p\varphi_p^0), \; w_0 = (t_f + t_c)V_e\varphi_e^0. \quad (19)$$

Note that $z(t_f) = y_e(t_f) - y_p(t_f)$, and the cost functional (7) can be rewritten as

$$J = |z(t_f)|^2 + \alpha \int_0^{t_f} u_p^2(t)dt - \beta \int_0^{t_f} u_e^2(t)dt. \quad (20)$$

Due to (14), the constraint (8) becomes

$$|w(t_f)| \le \mu_e a_e^{\max}. \quad (21)$$

Thus, the OPEG is reduced to the pursuit-evasion differential game for the system (15) – (16) with the cost functional (20) and the terminal evader's constrain (21). This game is called the Reduced Pursuit-Evasion Game (RPEG).

2.3. Saddle Points in Reduced Game

In this section, we obtain the pairs of strategies $(u_p^*(\cdot), u_e^*(\cdot))$, constituting the saddle point in the Reduced Game, i.e., satisfying for all admissible strategies $u_p(\cdot)$, $u_e(\cdot)$ the saddle point inequality

$$J(u_p^*(\cdot), u_e(\cdot)) \le J(u_p^*(\cdot), u_e^*(\cdot)) \le J(u_p(\cdot), u_e^*(\cdot)). \quad (22)$$

Let us define the values

$$s = 1 + \frac{1}{\alpha}\int_0^{t_f} h_p^2(t)dt - \frac{1}{\beta}\int_0^{t_f} h_e^2(t)dt > 0, \quad (23)$$

$$a = \frac{1}{\beta s}\int_0^{t_f} h_e(t)g_e(t)dt. \quad (24)$$

$$\nu_p = \frac{1}{\alpha}\int_0^{t_f} h_p^2(t)dt, \quad (25)$$

$$G_2 = \frac{1}{\beta}\int_0^{t_f} h_e(t)g_e(t)dt, \; G_3 = \frac{1}{\beta}\int_0^{t_f} g_e^2(t)dt, \quad (26)$$

$$d = \frac{\nu_p G_2^2}{G_1((s - \nu_p)G_3 + G_2^2)},$$

(27)

the matrix

$$G = \begin{bmatrix} s(0) & G_2 \\ -G_2 & G_3 \end{bmatrix}, \quad (28)$$

the vectors

$$b^+ = \begin{bmatrix} z_0 \\ w_0 - \mu_e a_e^{\max} \end{bmatrix}, b^- = \begin{bmatrix} z_0 \\ w_0 + \mu_e a_e^{\max}. \end{bmatrix}, \quad (29)$$

$$\omega_f^+ = (z_f^+, v_f^+)^T = G^{-1}b^+, \quad (30)$$

$$\omega_f^- = (z_f^-, v_f^-)^T = G^{-1}b^-, \quad (31)$$

and the sets

$$\Omega = \{(z_0, w_0) : |w_0 + az_0| < \mu_e a_e^{\max}\}, \quad (32)$$

$$\Omega^+ = \{(z_0, w_0) \notin \Omega : w_0 + az_0 \ge d\mu_e a_e^{\max}\}, \quad (33)$$

$$\Omega^- = \{(z_0, w_0) \notin \Omega : w_0 + az_0 \le -d\mu_e a_e^{\max}\}. \quad (34)$$

In what follows, we assume that the condition

$$\beta > \int_0^{t_f} h_e^2(t)dt \quad (35)$$

holds.

Remark 1. *Subject to the condition (35),*

$$\Omega^+ = \{(z_0, w_0) : w_0 + az_0 \ge \mu_e a_e^{\max}\}, \quad (36)$$

$$\Omega^- = \{(z_0, w_0) : w_0 + az_0 \le -\mu_e a_e^{\max}\}. \quad (37)$$

In this case, the planar sets Ω, Ω^+ and Ω^- do not intersect each other, and $\Omega \cup \Omega^+ \cup \Omega^-$ coincides with the entire (z_0, w_0)-plane.

The saddle point solutions of Reduced Game are defined separately for the cases $(z_0, w_0) \in \Omega$, $(z_0, w_0) \in \Omega^+$ and $(z_0, w_0) \in \Omega^-$.

Theorem 1. *If the condition (35) holds and $(z_0, w_0) \in \Omega$, the pair*

$$u_p^0(t) = -\frac{h_p(t)z_0}{\alpha s}, \; u_e^0(t) = \frac{h_e(t)z_0}{\beta s}, \quad (38)$$

is an open-loop saddle point in the Reduced Game.

Remark 2. *If $(z_0, w_0) \in \Omega$, then, the solution $w(t)$ generated by $u_e^*(\cdot)$ in (38), satisfies the inequality (21) strictly, i.e., $|w(t_f)| < \mu_e a_e^{\max}$.*

Theorem 2. *Let (35) hold and* $(z_0, w_0) \in \Omega^+$. *Then, the pair*

$$u_p^+(t) = -\frac{1}{\alpha} h_p(t) z_f^+,$$
$$u_e^+(t) = \frac{1}{\beta}\left[h_e(t) z_f^+ - g_e(t) v_f^+ \right] \quad (39)$$

is an open-loop saddle point in the Reduced Game.

Remark 3. *If* $(z_0, w_0) \in \Omega^+$, *then the optimal trajectory* $(z(t), w(t))$ *generated by the pair* $(u_p^+(\cdot), u_e^+(\cdot))$ *satisfies the terminal conditions*

$$z(t_f) = z_f^+, \ \ w(t_f) = \mu_e a_e^{\max}, \quad (40)$$

i.e., the terminal condition (21) is satisfied as an equality with the sign "+ ".

Theorem 3. *Let (35) hold and* $(z_0, w_0) \in \Omega^-$. *Then, the pair*

$$u_p^-(t) = -\frac{1}{\alpha} h_p(t) z_f^-,$$
$$u_e^-(t) = \frac{1}{\beta}\left[h_e(t) z_f^- - g_e(t) v_f^- \right] \quad (41)$$

is an open-loop saddle point in the Reduced Game.

Remark 4. *If* $(z_0, w_0) \in \Omega^-$, *then, the optimal trajectory* $(z(t), w(t))$ *generated by the pair* $(u_p^-(\cdot), u_e^-(\cdot))$ *satisfies the terminal conditions*

$$z(t_f) = z_f^-, \ \ w(t_f) = -\mu_e a_e^{\max}, \quad (42)$$

i.e., the terminal condition (21) is satisfied as an equality with the sign "− ".

3. SPECIAL CASE: FIRST-ORDER PURSUER AGAINST FIRST-ORDER EVADER

In this section, the theory of the previous section is applied to the particular case of (1) – (2) which is of a practical interest. This example illustrates some important features of the game solution.

3.1. Original Pursuit-Evasion Game

If both the pursuer and the evader have the first-order dynamics controller, then in the system (1) – (2), $n_p = 1$, $\bar{A}_p = -1/\tau_p$, $\bar{b}_p = 1/\tau_p$, $\bar{d}_p = 0$, $n_e = 1$, $\bar{A}_e = -1/\tau_e$, $\bar{b}_e = 1/\tau_e$, $\bar{d}_e = 0$, where τ_p and τ_e are the time constants of the pursuer's and the evader's controllers. The pursuer's and the evader's controls are the lateral acceleration commands.

In the OPEG, the controlled system is given by (3), where $x = (y_p, \dot{y}_p, a_p, y_e, \dot{y}_e, a_e)^T$,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1/\tau_p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1/\tau_e \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1/\tau_p \\ 0 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/\tau_e \end{bmatrix}.$$

The cost functional (7) becomes

$$J = (x_4(t_f) - x_1(t_f))^2 + \alpha \int_0^{t_f} u_p^2(t)dt - \beta \int_0^{t_f} u_e^2(t)dt.$$

The matrix Φ_e is

$$\Phi_e(t_f + t_c, t) =$$

$$\begin{bmatrix} 1 & t_f + t_c - t & -\tau_e^2 \psi((t_f + t_c - t)/\tau_e) \\ 0 & 1 & \tau_e[\exp(-(t_f + t_c - t)/\tau_e) - 1] \\ 0 & 0 & \exp(-(t_f + t_c - t)/\tau_e) \end{bmatrix},$$

where $\psi(t) = \exp(-t) + t - 1 \ge 0.$

Thus, the terminal inequality constraint (8) becomes

$$| x_4(t_f) + t_c x_5(t_f) - \tau_e^2 \psi(t_c / \tau_e) x_6(t_f) | \le \mu_e a_e^{\max},$$

where

$$\mu_e = \tau_e \int_{t_f}^{t_f + t_c} \psi((t_f + t_c - t)/\tau_e)dt =$$
$$\tau_e^2 (1 - \sigma + \sigma^2 / 2 - \exp(-\sigma)), \sigma = t_c / \tau_e. \quad (43)$$

3.2. Reduced Game

The scalar variables (13) and (14) become

$$z(t) = y_e - y_p + (t_f - t)(\dot{y}_e - \dot{y}_p) -$$
$$\tau_p^2 \psi((t_f - t)/\tau_p) a_p + \tau_e^2 \psi((t_f - t)/\tau_e) a_e,$$
$$w(t) = y_e + (t_f + t_c - t)\dot{y}_e +$$
$$\tau_e^2 \psi((t_f + t_c - t)/\tau_e) a_e.$$

The coefficient functions (17) in the differential equations (15) – (16) become

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$$h_p(t) = -\tau_p \psi((t_f - t)/\tau_p),$$
$$h_e(t) = \tau_e \psi((t_f - t)/\tau_e), \quad (44)$$
$$g_e(t) = \tau_e \psi((t_f + t_c - t)/\tau_e). \quad (45)$$

The differential equations (15) – (16) become

$$\dot{z} = -\tau_p \psi((t_f - t)/\tau_p) u_p + \tau_e \psi((t_f - t)/\tau_e) u_e, \quad (46)$$
$$\dot{w} = \tau_e \psi((t_f + t_c - t)/\tau_e) u_e. \quad (47)$$

The Reduced Game (RG) is formulated for the system (46) – (47) with the cost functional (20) and the terminal inequality constraint (21) where μ_e is given by (43).

3.3. Saddle Point Solutions

In this case, the solvability condition (35) reads

$$\beta > \beta^* = \tau_e^2 \int_0^{t_f} \psi^2((t_f - t)/\tau_e) dt. \quad (48)$$

In Fig. 2, the game space decomposition into the sets Ω, Ω^+ and Ω^- is shown for $t_f = 1$ s, $\nu = 0.9$, $a_e^{\max} = 100$ m/s^2, $\alpha = 0.05$, $\beta = 0.3$, $\tau_p = 0.2$ s, $\tau_e = 0.1$ s. For these parameters, $\beta^* = 0.2438$, and the solvability condition (48) is valid.

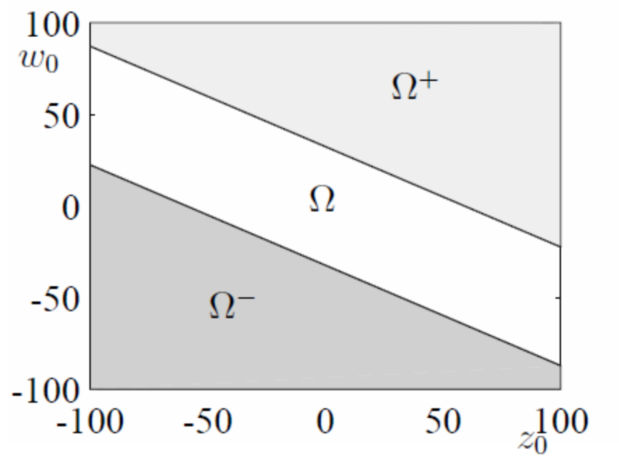

Figure 2: Game space decomposition

3.3.1. Solution for $(z_0, w_0) \in \Omega$

In this case, the optimal controls (38) are calculated by substituting $h_p(t)$ and $h_e(t)$ from (44) into the value of s. The solvability condition (35) yields $s > 0$.

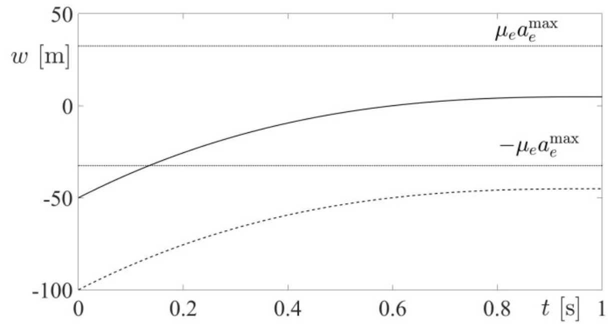

Figure 3: Solution for $(z_0, w_0) \in \Omega$

In this example, the terminal constraint (21) is $|w(t_f)| \le 32.5$. In Fig. 3, two optimal w-trajectories are shown for different initial conditions. If the game starts from $(z_0 = 100, w_0 = -50) \in \Omega$, then $w(t_f) = 4.895$ m satisfies the terminal inequality constraint (the trajectory is shown by the solid line). If the initial position is $(z_0 = 100, w_0 = -100) \notin \Omega$, $w(t_f) = -45.105$ m and the terminal constraint is violated (dashed-line trajectory). The straight lines $w = \pm \mu_e a_e^{\max} = \pm 32.5$ m depict the boundaries of the evader's constraint.

3.3.2. Solution for $(z_0, w_0) \notin \Omega$

We continue using the same parameters as in the previous subsection. In this example, $a = 0.92$, $\mu_e a_e^{\max} = 32.5$, the matrix (28) is

$$G = \begin{bmatrix} 3.72 & 2.04 \\ -2.04 & 5.91 \end{bmatrix}.$$

For $(z_0, w_0) = (100, 50) \in \Omega^+$,

$$b^+ = \begin{bmatrix} 100 \\ 172.5 \end{bmatrix}, \omega_f^+ = \begin{bmatrix} z_f^+ \\ v_f^+ \end{bmatrix} = \begin{bmatrix} 21.22 \\ 10.29 \end{bmatrix}.$$

For $(z_0, w_0) = (-100, -20) \in \Omega^-$,

$$b^- = \begin{bmatrix} -100 \\ 12.5 \end{bmatrix}, \omega_f^- = \begin{bmatrix} z_f^- \\ v_f^- \end{bmatrix} = \begin{bmatrix} -23.56 \\ -6.02 \end{bmatrix}.$$

In Figs. 4 – 5, the optimal trajectories, generated by the saddle-point pairs $(u_p^+(\cdot), u_e^+(\cdot))$ and $(u_p^-(\cdot), u_e^-(\cdot))$, are shown ($z(t)$ and $w(t)$ in Figs. 4 and 5, respectively). It is seen that under the controls $(u_p^+(\cdot), u_e^+(\cdot))$, $z(t_f) = z_f^+ = 21.22$ m and $w(t_f) = \mu_e a_e^{\max} = 32.5$ m, i.e., the terminal equality conditions (40) are satisfied. Correspondingly, under the controls $(u_p^-(\cdot),u_e^-(\cdot))$, $z(t_f)=z_f^-=-23.56$ m and $w(t_f)=-\mu_e a_e^{\max}=-32.5$ m, i.e., the terminal equality conditions (42) are satisfied.

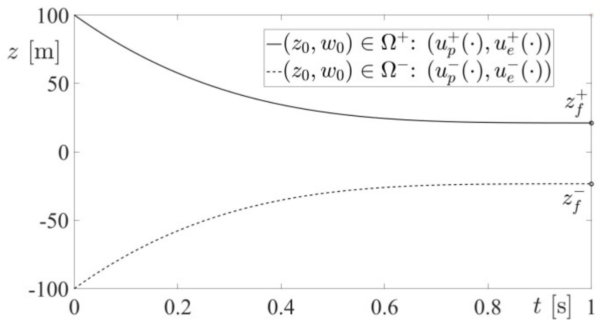

Figure 4: Optimal z-trajectories

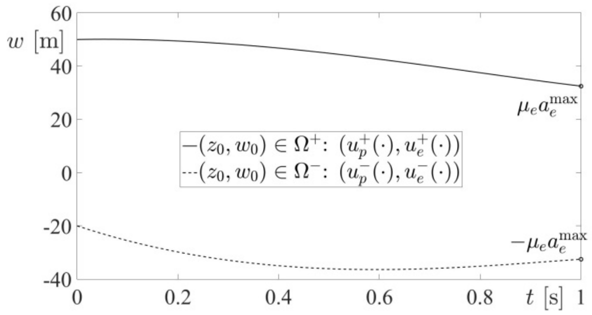

Figure 5: Optimal w-trajectories

The optimal controls $u_p(t)$ and $u_e(t)$ are depicted in Figs. 5 and 6, respectively.

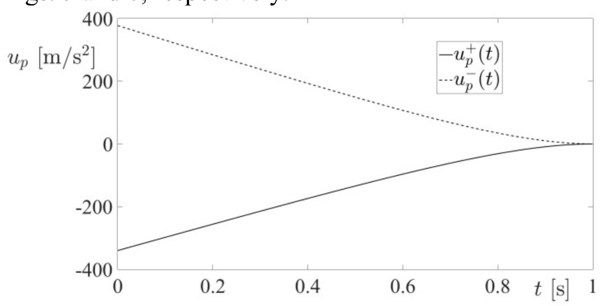

Figure 6: Optimal controls $u_p(t)$

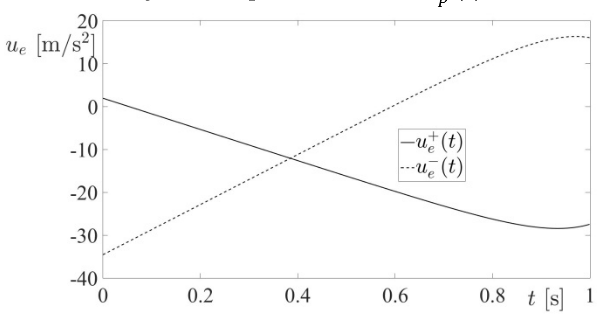

Figure 7: Optimal controls $u_e(t)$

Due to Theorem 2, the pair $(u_p^+(\cdot),u_e^+(\cdot))$ given by (39) constitutes the saddle point in the Reduced Game if and only if $(z_0,w_0)\in\Omega^+=\{w_0+0.92z_0\geq 32.5\}$ (see Fig. 2). Similarly, for $(z_0,w_0)\in\Omega^-=\{w_0+0.92z_0\leq -32.5\}$ (see Fig. 2), the saddle point in the Reduced Game is $(u_p^-(\cdot),u_e^-(\cdot))$ given by (39).

Table 1: Results for $(z_0,w_0)\notin\Omega$

$(z_0,w_0)\in\Omega^+$		$(z_0,w_0)\in\Omega^-$	
Controls	Result	Controls	Result
$(u_p^+(\cdot),u_e^+(\cdot))$	1939.2	$(u_p^-(\cdot),u_e^-(\cdot))$	2488.2
$(u_p^+(\cdot),u_e^-(\cdot))$	418.8	$(u_p^-(\cdot),u_e^+(\cdot))$	1463.1
$(u_p^-(\cdot),u_e^+(\cdot))$	2347.7	$(u_p^+(\cdot),u_e^-(\cdot))$	2836.7

Let us chose the initial position $(z_0,w_0)=(100,50)\in\Omega^+$ and calculate the cost functional (20) for three pairs of control functions: for $(u_p^+(\cdot),u_e^+(\cdot))$, for $(u_p^+(\cdot),u_e^-(\cdot))$ and for $(u_p^-(\cdot),u_e^+(\cdot))$. For $(z_0,w_0)=(-100,-20)\in\Omega^-$ we calculate (20) for $(u_p^-(\cdot),u_e^-(\cdot))$, for $(u_p^-(\cdot),u_e^+(\cdot))$ and for $(u_p^+(\cdot),u_e^-(\cdot))$. The results are presented in Table 1. It is seen that for $(z_0,w_0)\in\Omega^+$, the saddle point inequality (22) with $u_p(\cdot)=u_p^-(\cdot)$, $u_e(\cdot)=u_e^-(\cdot)$ is satisfied for $(u_p^+(\cdot),u_e^+(\cdot))$. For $(z_0,w_0)\in\Omega^-$, the saddle point inequality (22) with $u_p(\cdot)=u_p^+(\cdot)$, $u_e(\cdot)=u_e^+(\cdot)$ is satisfied for $(u_p^-(\cdot),u_e^-(\cdot))$.

3.3.3. Feedback realizations of optimal strategies

The complete solution of the Original Game in the class of feedback strategies is the topic of the future research. However, in this paper, we propose the following feedback realization of the saddle point strategies (38), (39) and (41). This realization is based on implementing the open-loop strategy where a current position $(t,z(t),w(t))$ is used instead of the initial position $(0,z_0,w_0)$. The idea of constructing a feedback control based on an open-loop strategy is well known in the control literature (see e.g., (Gabasov, Gaishun, Kirillova, and Prishchepova 1992)).

For $(z_0,w_0)\in\Omega$, the feedback realization of (38) is

$$u_p^0(t,z) = -\frac{h_p(t)z}{\alpha s(t)},\; u_e^0(t,z) = \frac{h_e(t)z}{\beta s(t)}, \qquad (49)$$

where

$$s(t) = 1 + \frac{1}{\alpha}\int_t^{t_f} h_p^2(t)dt - \frac{1}{\beta}\int_t^{t_f} h_e^2(t)dt. \qquad (50)$$

In order to construct the feedback realization for $(z_0, w_0) \notin \Omega$, let us define the matrix

$$G(t) = \begin{bmatrix} s(t) & G_2(t) \\ -G_2(t) & G_3(t) \end{bmatrix}, \qquad (51)$$

where

$$G_2(t) = \frac{1}{\beta}\int_t^{t_f} h_e(t) g_e(t)dt,$$

$$G_3(t) = \frac{1}{\beta}\int_t^{t_f} g_e^2(t)dt, \qquad (52)$$

and the vectors

$$b^+(z,w) = \begin{bmatrix} z \\ w - \mu_e a_e^{\max} \end{bmatrix},$$

$$b^-(z,w) = \begin{bmatrix} z \\ w + \mu_e a_e^{\max}. \end{bmatrix}, \qquad (53)$$

$$\omega_f^+(t,z,w) = (z_f^+(t,z,w), v_f^+(t,z,w))^T = G^{-1}(t) b^+(z,w), \qquad (54)$$

$$\omega_f^-(t,z,w) = (z_f^-(t,z,w), v_f^-(t,z,w))^T = G^{-1}(t) b^-(z,w). \qquad (55)$$

Then, for $(z_0, w_0) \in \Omega^+$ and $(z_0, w_0) \in \Omega^-$, (39) and (41) become

$$u_p^+(t,z,w) = -\frac{1}{\alpha} h_p(t) z_f^+(t,z,w),$$

$$u_e^+(t,z,w)) = \frac{1}{\beta}\left[h_e(t) z_f^+(t,z,w) - g_e(t) v_f^+(t,z,w)\right], \qquad (56)$$

and

$$u_p^-(t,z,w) = -\frac{1}{\alpha} h_p(t) z_f^-(t,z,w),$$

$$u_e^-(t,z,w)) = \frac{1}{\beta}\left[h_e(t) z_f^-(t,z,w) - g_e(t) v_f^-(t,z,w)\right], \qquad (57)$$

respectively.

We remind that in this paper, we do not present a strict theoretical justification of feedback solutions (49), (56) and (57).

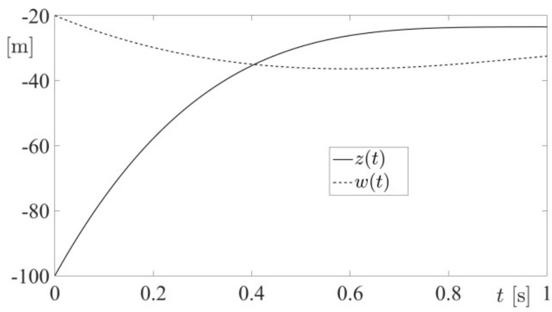

Figure 8: Trajectories generated by (57)

In Fig. 8, the trajectories $z(t)$ and $w(t)$ generated from the position $(z_0, w_0) = (-100, -20) \in \Omega^-$ by the feedback strategies (57), are depicted for the same parameters as in the previous section. These trajectories are close to those generated by the saddle point open-loop strategies (see Figs. 4 and 5). The terminal values are $z(t_f) = -23.484\,\text{m} \approx z_f^- = -23.56\,\text{m}$, $w(t_f) = -32.5\,\text{m} = -\mu a_e^{\max}$.

Now, we examine the behavior of the feedback strategies in the case of noisy state measurements. At each step t_i of the numerical solution, the true values $z_i = z(t_i)$ and $w_i = w(t_i)$ are replaced by $\tilde{z}_i = z_i + \eta_{zi}$ and $\tilde{w}_i = w_i + \eta_{wi}$, where $\eta_{zi} \sim U[-\delta_z, \delta_z]$ and $\eta_{wi} \sim U[-\delta_w, \delta_w]$ are the uniformly distributed measurement errors for z and w respectively. Two cases are distinguished: (I) both the pursuer and the evader obtain the noised state information, and (II) the pursuer obtains the noise information, whereas the evader uses accurate measurements. Let us denote $z_I(t)$, $z_{II}(t)$ and $w_I(t)$, $w_{II}(t)$ the trajectories $z(t)$ and $w(t)$ in the cases (I) and (II), respectively. In Fig. 9, the differences $\Delta_I^z(t) = |z(t) - z_I(t)|$ and $\Delta_{II}^z(t) = |z(t) - z_{II}(t)|$ are shown for $\delta_z = \delta_w = 50$ m. In this simulation, $z_I(t_f) = -23.56$ m, $z_{II}(t_f) = -24.46$ m, yielding $\Delta_I^z(t_f) = 0.46\,\text{m}$, $\Delta_{II}^z(t_f) = 0.98\,\text{m}$. Thus, the terminal error is larger in the case where the evader has an information advantage.

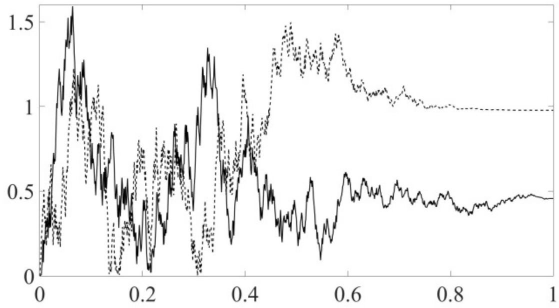

Figure 9: Differences $\Delta_I^z(t)$ and $\Delta_{II}^z(t)$

In Fig. 10, the differences $\Delta_I^w(t) = |w(t) - w_I(t)|$ and $\Delta_{II}^w(t) = |w(t) - w_{II}(t)|$ are shown. It is seen that in the case (I) the difference is large, whereas in the case (II), the difference is close to zero. In this simulation, $w_I(t_f) = -28.21$ m, $w_{II}(t_f) = -32.5$ m, yielding $\Delta_I^w(t_f) = 4.285$ m, $\Delta_{II}^w(t_f) = 0$ m.

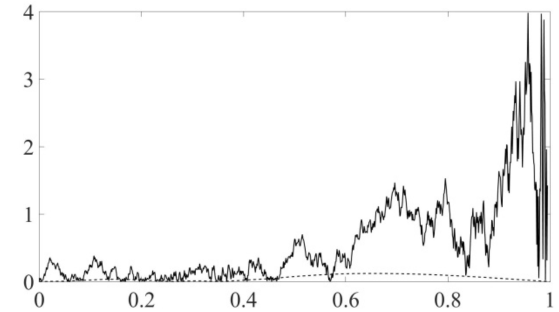

Figure 10: Differences $\Delta_I^w(t)$ and $\Delta_{II}^w(t)$

4. CONCLUSIONS

The practically important special case of a linear-quadratic differential game with a terminal inequality constraint was considered. This game models a pursuit of an evader with two objectives: (i) maximizing the cost functional, and (ii) hitting a stationary target. In this case, both the pursuer and the evader have first-order controller dynamics. The case was treated based on the results presented in the previous paper of the authors and outlined briefly in this paper. In the special case,

- the solvability condition was established;
- the game space decomposition into three non-intersecting sets was constructed;
- the saddle point game solutions were derived;
- the fulfilment of the saddle point inequality was demonstrated;
- the feedback realization of the saddle point open-loop strategies was presented and simulated with state measurement errors.

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