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Train Schedule Analysis and Optimization with the Max-Plus Automaton

Alexander Edthofer[1,*], Martin Bicher[2] and Felix Breitenecker[1]

[1]Institute of Analysis and Scientific Computing, Vienna University of Technology, Wiedner Hauptstraße 8-10, 1040 Vienna, Austria
[2]dwh Gmbh, Neustiftgasse 57-59, 1070 Vienna, Austria

[*]alexander.edthofer@tuwien.ac.at

Abstract

This work deals with methods to improve efficiency of train networks and applies them on the example of Austrian freight train traffic. The aim is to analyze the utilization of the capacity of the tracks and, if possible, to optimize them. We use implementations of the Max-Plus automaton and apply them for given freight train traffic data of a small rail network within Vienna, the capital city of Austria. The theoretical background is the Max-Plus algebra, with which the Max-Plus automaton is described. This model deals in general with microscopic networks. In this paper the Max-Plus automaton is applied to a schedule of freight trains, which run in the southwest of Vienna. It shows, that the efficiency of the utilization of the tracks can be increased, if the sequence of the trains is changed.

Keywords: Max-Plus algebra, transport optimization, freight transport, train simulation

1. Introduction

In the past decades, train traffic in passenger and freight transport has increased enormously around the world. From 1998 to 2018 train traffic in Austria has increased from 179 to 310 million passengers per year (UNECE, 2020). Similiar, with austrian freight traffic, the total mass of transported goods has increased from 73 to 105 million tons (UNECE, 2020). In order to handle the increasing demand, new possibilities have to be invented.

This can be achieved by building new tracks, expanding the existing ones, or simply making better use of them. The latter is obviously the best economical option. In order to increase the efficiency, the capacity and the load on the network have to be considered.

With Max-Plus algebra, models can be developed that are used in particular to analyse and optimise event-discrete processes, such as in production chains or timetables (Butkovič, 2010). This work deals with the latter and therefore a model applying the Max-Plus algebra is presented in detail.

The Max-Plus automaton, as described by Nikola Bešinović and Rob M.P. Goverde in (Borndörfer et al., 2018), deals with the sequence of individual trains on a certain railway network. The travel times of the individual trains are written in square matrices, the size of which is equal to the number of tracks. The sequence of the moves can be carried out accordingly by multiplying these matrices. The result can then be used to determine the capacity occupation on the track area. That is the amount of time, that train paths block the tracks.

In this contribution this model is applied on a section of the freight train schedule between the two stations Wien Westbahnhof and Wien Hauptbahnhof. The data

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made available are part of the *A&O* project with the Österreichische Bundesbahnen AG. The capacity load for this dataset was calculated by MATLAB using the given timetables. The goal is to reduce the capacity load by swapping individual trains.

2. State of the art

The most common method in Europe to analyze the capacity assessment of railway timetables is called timetable compression method. This method shifts train paths, such that the time difference between two paths, that provides conflict-free runs of the trains, is minimal. But one disadvantage is the modelling of nodes, for expample switch areas. The timetable compression method underestimates the capacity occupation in this issue (Lindner, 2011).

An analytic model for capacity assessment of nodes is the Max-Plus automaton. The model is described and introduced with a small example by Nikola Bešinović and Rob M.P. Goverde in (Borndörfer et al., 2018) and also shortly in (Heidergott et al., 2014). However, it is not applied on a real timetable, which is presented in this work.

3. Max-Plus algebra

The Max-Plus algebra provides the mathematical basics for the implemented model. As mentioned by Heidergott et al. (2014), the Max-Plus algebra $\mathcal{R}_{max} = (\mathbb{R}_{max}, \oplus, \varepsilon, \otimes, 0)$ is a commutative half ring with a neutral element over the body $\mathbb{R}_{max} := \mathbb{R} \cup \{\varepsilon\}$. For the variable ε applies $\varepsilon := -\infty$. For $a, b \in \mathbb{R}_{max}$ the operations $\oplus$ and $\otimes$ are defined as follows,

$$a \oplus b := \max\{a, b\},$$
$$a \otimes b := a + b,$$

where max and + are the conventional operations known from real numbers.

As a commutative half ring with a neutral element, the operators $\oplus$ and $\otimes$ are associative as well as commutative and have the neutral elements ε and 0 respectively. Furthermore, the operation $\otimes$ is distributive with respect to $\oplus$. Apart from that, $\oplus$ is idempotent and ε is absorbing with respect to $\otimes$.

The concept of potency can now be redefined within Max-Plus algebra. The symbol $\cdot$ denotes the conventional multiplication within the real numbers.

$$a^{\otimes b} = \underbrace{a \otimes a \otimes \ldots \otimes a}_{b \text{ times}} = b \cdot a.$$

The Max-Plus algebra can also be expanded on vectors and matrices in the following sense. Be $\mathbb{R}_{max}^{n \times m}$ the set of all $n \times m$ matrices with values in $\mathbb{R}_{max}$. The operations can be taken analogously. For simpler representation, variables with underscores, such as $\underline{n}$, are defined as the set of all numbers up to itself $\underline{n} := \{1, 2, \ldots, n\}$.

$$[A \oplus B]_{ij} := a_{ij} \oplus b_{ij} = \max\{a_{ij}, b_{ij}\},$$
$$\forall A, B \in \mathbb{R}_{max}^{n \times m} \quad i \in \underline{n}, j \in \underline{m}.$$

The maximum of the respective entries is thus calculated component by component.

$$[A \otimes B]_{ik} := \bigoplus_{j=1}^{l} a_{ij} \otimes b_{jk} = \max_{j \in \underline{l}}\{a_{ij} + b_{jk}\},$$
$$\forall A \in \mathbb{R}_{max}^{n \times l}, B \in \mathbb{R}_{max}^{l \times m} \quad i \in \underline{n}, j \in \underline{l}, k \in \underline{m}.$$

$$A^{\otimes b} := \underbrace{A \otimes A \otimes \ldots \otimes A}_{b \text{ times}},$$
$$\forall A \in \mathbb{R}_{max}^{n \times n}, b \in \mathbb{R}.$$

These operations have as well neutral elements. An $n \times m$ matrix, whose entries are all equal to ε, is neutral with respect to $\oplus$ and is denoted by $\mathcal{E}(n, m)$. The matrix $E(n, m)$, defined by

$$[E(n, m)]_{ij} := \begin{cases} 0 & i = j, \\ \varepsilon & \text{sonst}, \end{cases}$$

is the neutral element with respect to $\otimes$. For $n = m$, $E(n, n)$ is also called the unit matrix.

By using $\oplus$ component by component, most of the properties of $\mathcal{R}_{max}$ are transferred to $\mathcal{R}_{max}^{n \times n}$. Only the operation $\otimes$ is not commutative. Since this property is missing, $\mathcal{R}_{max}^{n \times n}$ is called a half-ring with neutral element, because apart from that, the attributes are adopted analogously from $\mathcal{R}_{max}$.

4. Max-Plus automaton

A planned rail timetable can be modeled as a discrete event dynamic system (DEDS) using the Max-Plus algebra. This model is an irreducible first-order model of the form

$$x(k) = A \otimes x(k-1) \oplus B \otimes u(k).$$

In the following the Max-Plus automaton is presented, with which the infrastructure resources and the blocking of these by trains can be represented. Hence the capacity load on the resources can be calculated accordingly (Borndörfer et al., 2018).

4.1. Structure of the model

A Max-Plus automaton is a tuple $H = (T, R, M, s, f)$. Hereby T describes the set of all trains that can block resources. The resources are the tracks and are described as a set by R. The mapping $M : T \to \mathbb{R}_{max}^{|R|\times|R|}$ is defined as in (1). It assigns every train a matrix.

$$[M]_{ij}(l) = \begin{cases} 0 & \text{for } i = j, i \notin R(l), \\ f_j(l) - s_i(l) & \text{for } i, j \in R(l), \\ \varepsilon & \text{else.} \end{cases} \quad (1)$$

The start time of train l on resource i is described by $s_i(l)$. The value $f_i(l)$ describes the time, when the track i is left by train l. These values are combined to the $|R|$-dimensional vectors $s(l) \in \mathbb{R}_{max}^{|R|}$ and $f(l) \in \mathbb{R}_{max}^{|R|}$. An entry in the matrix M as defined in (1) thus stands for the period of time between leaving resource j and starting i, if the train l uses i and j.

A schedule $w = l_1 \cdots l_n$, with $l_1, \ldots, l_n \in T$, describes an ordered sequence of moves. The load on all train lines is characterized by $M(w)$, which can be calculated by

$$M(w) = M(l_1 \cdots l_n) = M(l_1) \otimes \ldots \otimes M(l_n).$$

In order to determine the capacity load $\mu(w)$ of a timetable w, an upper limit $x(w)$ of this is required, where $x(w) = M(w) \otimes x(e)$. The entries of the vector $x(e) \in \mathbb{R}_{max}^{|R|}$ are all equal to 0. Since we are always examining periodic timetables, we add the first train at the end of w, since the start time of this symbolizes the end of the capacity load. That first train will be designated with a. The calculation of $\mu(w)$ is now done as in (2). The minimal component of the already calculated upper limit is taken, but as we look at periodic timetables, the travel time of the first train of the next period has to be subtracted.

$$\mu(w) = \min_{i \in R(a)} \left(x_i(wa) - (f_i(a) - s_i(a)) \right) \quad (2)$$

Since the Max-Plus automaton allows the trains to run just gradually and thus binds them to the track sections in a given order, additional models are required to represent overtaking, changing passengers or train coupling or decoupling, which is not considered here (Borndörfer et al., 2018).

4.2. Freight train traffic in the southwest of Vienna

This model is now used to analyze a timetable between Wien Westbahnhof and Wien Hauptbahnhof. This contains 21 freight trains that travel on 15 different tracks, so it is only a small extract of the actual plan. The data is available as part of the *A&O* project with the Österreichische Bundesbahnen AG.

The plan is initially included as a table in MATLAB and then the individual columns required are saved as vectors. In this way, the start and end times of the individual trains on the respective tracks can be determined, whereby it is assumed that each train needs about five seconds after arriving on the next track until it has left the previous track section. Then the mapping matrix M can be calculated. In table 1 we can see a part of such a matrix, for example for the first train, which starts at Wien Westbahnhof (track 1) and arrives at Wien Hauptbahnhof (track 12). The first entry describes, that the train needs 5 seconds or 0.0833 minutes for leaving the first track after he started. The entry (1, 12) describes, as the twelfth track is the last one, that the train drives 15 minutes and 5 seconds in total on the resources. As the entries in the columns 13, 14 and 15 are all $-\infty$, these three tracks are not used by the first train.

Table 1. Matrix of the first train, in which the times between the stations are shown.

	1	2	3		12	13	14	15
1	0.0833	1.0833	3.0833	. . .	15.0833	-Inf	-Inf	-Inf
2	-0.9167	0.0833	2.0833	. . .	14.0833	-Inf	-Inf	-Inf
	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮
15	-Inf	-Inf	-Inf	. . .	-Inf	-Inf	-Inf	0

As described in section 4.1, we calculate the capacity load of the route section, which is $\mu_1 = 466.75$, with minutes selected as the unit.

Furthermore it is interesting to compare the load if, for example, the order of two trains is reversed. This is achieved by swapping two columns of the matrices in the code, each of which contains the start and end times of the according trains. For example, the capacity load ($\mu_2 = 442.75$) can be reduced by 24 minutes if the trains 19 and 20 change order. We will henceforth denote the original sequence of trains as variant 1, the sequence with swapped trains 19 and 20 as variant 2.

The individual routes of the trains can be graphically illustrated, as seen in Figure 1. The circles $\circ$ show the individual stations, the arrows $\to$ describe the way of the respective train. Train 18 starts from the station Wien Westbahnhof and ends at the station Wien Hauptbahnhof.

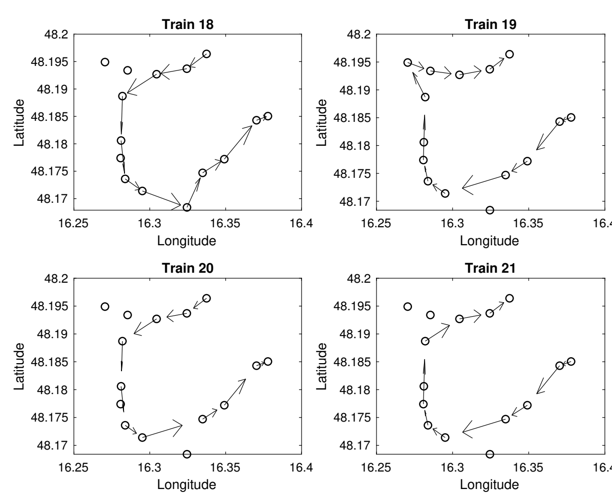

Figure 1. Course of the trains 18–21

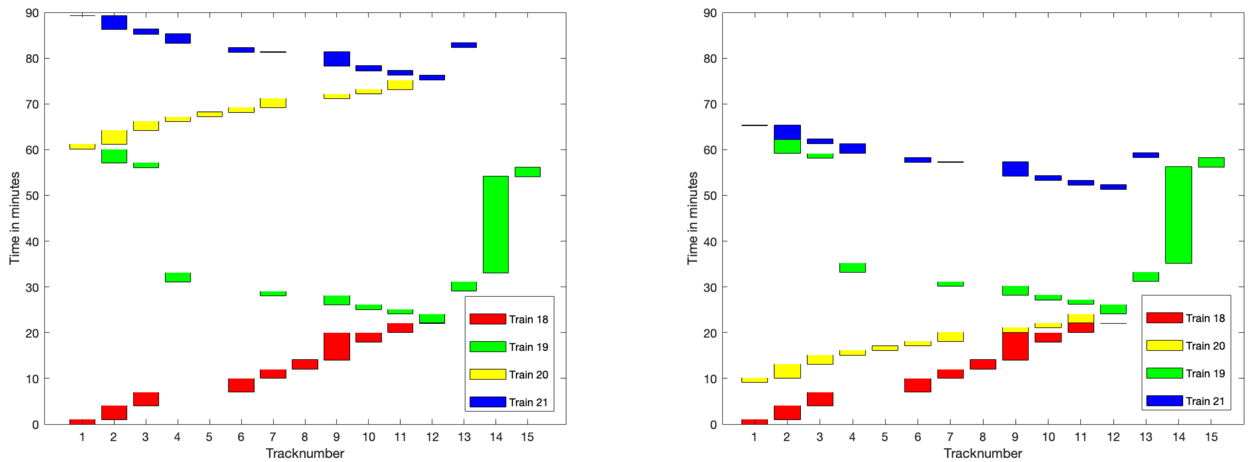

Figure 2. Comparison of the times of trains – variant 1 (left) and variant 2 (right)

Variant 1 now denotes the sequence to which the capacity load μ_1 belongs, analogously for variant 2 and the value μ_2, i.e. for the reversed sequence of train 19 and train 20. The different values of μ_1 and μ_2 are particularly evident in the pictures in figure 2. Due to the changed order of train 19 and train 20, neighboring trains now have similar routes, which means that the trains can start earlier. Hence the difference of 24 minutes results. This difference can be seen by the fact that train 21 in variant 1, depicted in the left picture in figure 2, leaves track 1 at a time of 89 minutes, but in variant 2, shown in the right picture in figure 2, already at 65 minutes. This indicates that variant 2 is the better one, since the time, when trains block resources is lowered.

5. Conclusion

In this work, the Max-Plus algebra and its application for train schedule analysis and optimization were presented. In particular, the capacity assessement of nodes is evaluated.

The computational effort displays, that the Max-Plus automaton is only suitable for small networks or timetables, if it can reproduce them in detail. The infrastructure, such as individual routes and trains, can be easily embedded in the model, whereby a short period of the timetable is advantageous.

The work can now be expanded or deepened in a few aspects. In the application example of the Max-Plus automaton, a possible improvement of the timetable was presented in order to reduce the capacity assessement. The investigated sequence of variant 2 was an improvement, but it probably was not the optimum of the train schedule. This can of course be optimized, however $(n-1)!$ possibilities must be executed, whereas n is the number of trains to be examined. These were in this case study $2.432902 \cdot 10^{18}$ different cases. The calculation of one case takes approximately 0.1 seconds, so the calculation of all cases would last around $7.7147 \cdot 10^{9}$ years. This fact hints, that the Max-Plus automaton can improve the capacity load, but even for the small extent of 21 trains, as in this case study, it is nearly impossible to find the optimum. Therefore only two cases were considered here. This shows, that the model is not very suitable for optimization of systems of larger dimensions, but at least can improve them.

In a future contribution the idea is to compare the introduced approach with another modelling approach as described in Heidergott et al. (2014), which is also based on the Max-Plus algebra. They introduce a method to map larger networks with more conditions, called Performance Evaluation of Timed Events in Railways ("PETER").

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