



The synthesis of the control function in optimal tasks as a N-dimensional area using parallel projection on 2D plane

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Abstract

Solving problems of multi-criteria decision analysis for Decision Support Systems (DSS) we need to proceed the complex system of N non-linear equations and inequalities which describe the solution feasible region. It means that we have to solve a heavy algebraic problem wasting a lot of computing resources. The best way to get the optimal decision is a describing the algebraic system in geometrical terms for graphical solving. The paper describes possible theoretical and program media for such tasks. The purpose is the synthesis of the optimal control function representation in N-dimensional space using its 2D projections in problems of Multi-Criteria Decision Analysis (MCDA) and Multi-Criteria Decision Making (MCDM).

Keywords: Pareto; Clifford; projection; geometric algebra; MCDA; MCDM

1. Introduction

Now the most important problem in all stages of a complex organization and technical system (COTS) life cycle (Sokolov et al., 2015) is a multi-criteria estimation task of management control effectiveness (Krasnoshchekov et al. 1979, Avduevskii et al. 1988, Tsvirkun and Akindiev, 1993). The prospective solution is based on dynamic interpretation of MCDA and COTS proactive management program technology. In that case the common algorithm of a Pareto set construction and research (the set of nondominated alternatives) for the studied class of problems of efficiency theory includes next base steps:

Step 1. Formation, analysis and interpretation of input

data used for the synthesis of COTS multi-structural macrostates. Construction and correction of description of models used for structural and functional synthesis of the COTS appearance and the COTS control system.

Step 2. Planning of the problem solution of the synthesis of COTS multi-structural macrostates. Determination of time and resources costs needed for the discussed problem of analysis and synthesis of COTS proactive management effective variants.

Step 3. Construction and approximation of feasible goals sets for logic-dynamic models describing COTS proactive management which implicitly determines COTS appearance variants (COTS multi-structural macrostates). Papers (Chernous'ko 1988, Dolgui et al. 2019) discuss it in detail.



Step 4. The set of COTS appearance requirements (particular indicators of COTS management effectiveness) are orthogonally projected on feasible goals sets of logic-dynamic models (Figure 1). As result (Dolgui et al. 2019, Ivanov and Sokolov 2019) the set of inconclusive decisions (also called the set of

nondominated alternatives, the Pareto set) is formed.

Step 5. Formation and interpretation of output data. Results are presented in some convenient form, for example, for adaptive program development of COTS progress proactive management and its corresponding impacts providing stable enough plans realization.

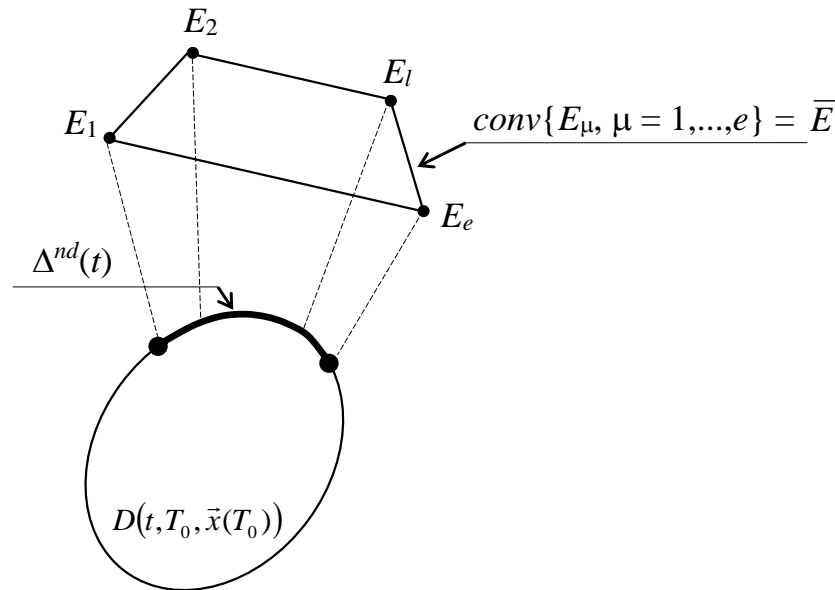


Figure 1. Construction of a Pareto set (the set of nondominated alternatives). The set is selected by the bold line

We propose to use well-known methods of projective geometry, geometric algebra, Clifford algebra and geometric programming for realization this generalized algorithm. Before these methods discussions in multi-criteria estimation tasks of COTS management control effectiveness, let us shortly observe current ways of such problem solutions.

2. State of the art

There are a lot of methods for MCDA problems solving. The main idea of all methods is a simplification of final optimization expressions to provide for experts the best way to select the right decision using an appropriate set of criteria. Now we try to observe some popular methods and compare them with our method of projections. All methods are realized as software applications because the MCDA problem for DSS needs a lot of calculation resources.

Some program solutions are based on an adaptive scalarization method (Eichfelder 2009) which aims to contract n criteria functions to one scalar function described the ideal system state. It must be led to the minimum or maximum for the optimal behavior of the management control function. Some trouble is in searching such function joining criteria of different dimensions. Also, this method squeezes the solution visualization for expert. We do not criticize this method but need to note some nuances of its use.

Similar methods are also based on the Multi Attribute Utility Theory (Bragge et al. 2010). The right decision is being searched for the expectation maximum of some utility function. It shares decision areas for expert analysis because the selection of the right solution is based on the maximal utility function value expectation. The utility function tries to describe all preferences of selected cases. Using this method requires fixing two problems of correctly constructing the utility function and adjusting the expected value.

Multi-objective Combinatorial Optimization (MOCO) is realized with the ONLINEMOCO web service (Ceyhan et al. 2019). Firstly, some nondominated points are found in the highest coverage gaps with Subset Based Algorithm. Then Territory Defining Algorithm requires a threshold coverage gap value to be satisfied by the final nondominated point set and uses this threshold to reduce the space searched by the algorithm. The method needs using two algorithms and does not guarantee the concrete solution.

Some popular methods are united in the ELECTRE family of MCDA algorithms. For example, IRIS and VIP (Kumar et al. 2017) can be used for risk analysis in the situation of robustness fixing or multicriteria performances aggregating. Interactive Robustness analysis and parameters' Inference for multicriteria Sorting problems (IRIS) is specified when the decision maker cannot determine weights of criteria, but supplies an example assignment. The Variable

Interdependent Parameters (VIP) method uses different levels of detail to reflect each criterion significance.

Rubis is a perspective method to select better alternatives from pairs and finally to choose the best one (Bisdorff et al. 2008). Whereas IRIS sorts alternatives into ordered classes based on outranking relation, Rubis is based on bipolar-valued outranking relation. It needs more time to make decision, but inspires better choice for experts. The Preference Ranking Organization for Enrichment of Evaluations (PROMETHEE) is similar to ELECTRE's Rubis because it uses pairwise comparisons to range alternatives and criteria, but provides better visualization of selection and can be combined with other MCDM methods (Zha et al. 2020).

The General Regression with Intensities of Preference (GRIP) method ranks a finite set of actions evaluated on multiple criteria (Figueira et al. 2009). In the first stage, GRIP collects input data, criteria and actions. In the second stage, a decision maker gives preference information for actions sorting. In the third stage, conditions are linear constructed and applied to value functions. In the fourth stage, the decision maker estimates value functions compatibility with given preference information and decides to accept the solution or change inconsistent preference information and restart with stage 2.

The TODIM method is based on empirical evidence

Table 1. The MCDM methods comparison

Method	Calculation complexity	Expert participation	Visualization
Adaptive scalarization method	Medium	Low	Low
Multi Attribute Utility	High	Low	Low
MOCO	High	Low	Medium
IRIS	Medium	Medium	Low
GRIP	Medium	High	High
PROMETHEE	High	Low	High
VIKOR	High	Low	Low
Feasible Goals Method	High	High	High
The method of parallel projections (proposed here)	Low	High	High

Methods of projective geometry could be used for making the shape image on the 2D plane. A practical mission of projective geometry is a graphical visualization of 3D in computer applications. There are a lot of variants to correspond an object to its projection in another space (Lehmer, 2005). Technical drawing uses the parallel projection to image the 3D thing onto the 2D plane. Two 2D projections onto the each to other perpendicular planes are enough for the full graphic description of this thing. In the case of more than 3 dimensions we can use projections onto the hyperplane or $N-1$ 2D planes, where N is a number of space dimensions (Philippov et al., 1986).

The practical targets of the projection using are

- the looking for the coordinates of points on the

of psychology how decisions are effectively accepted in high risk situations (Kazancoglu and Burmaoglu, 2013). The TODIM global value function aggregates all criteria values that makes possible to search the maximum of this function. The global function unites qualitative and quantitative data and provides the objective decision in result.

VIKOR focuses on ranking and sorting alternatives in the presence of conflicting criteria (Sajja, 2013). An algorithm based on VIKOR calculates best and worst values of criteria functions and then seeks the compromise solution. The compromise can be evaluated with Manhattan and Chebyshev distances between the best and the worst values. It lets to resolve conflicts in the situation of big extremities.

The method discussed in this paper was inspired by Lotov's Feasible Goals Method (Lotov et al. 2001). The appropriate solutions set can be expressed geometrically in criteria space as a convex function. Physically it is a closed form included the Pareto set where the right decision can be selected. The decision can be estimated by the expert on the flat diagram of slices or projections of this form with a common coordinate system. Observing overlapped areas of possible solutions (so-called the decision map), the expert makes choice. But using analytical geometry to construct a convex form complicates the task. Several MCDM methods are compared in Table 1.

surface of the ND object;

- the measurement of the distances between points on the surface of the ND object, the calculation of the shortest way between them on the surface;
- the determination of the polyline of the intersection of two or more ND objects.

The polyline of the intersection of several ND objects consists extreme points of the control function. For instance, some process is described by the function of two variables. It means that such process in terms of the analytical geometry can be expressed as a 3D surface. The equation determining this surface is

$$f_1(x_1, x_2, x_3) = const, \quad (1)$$

where x_1, x_2, x_3 are Cartesian coordinates of the surface

points.

Let approximate this surface by a simple shape. It can be a cone or some else. In the same time, we have some conditions in the analytical form which can be also described as a 3D shape (a cylinder, a sphere, etc.) Its equation is

$$f_2(x_1, x_2, x_3) = const. \quad (2)$$

Thus, the intersection polyline equation is

$$p(x_1, x_2, x_3) = f_2(x_1, x_2, x_3) \cap f_2(x_1, x_2, x_3). \quad (3)$$

This polyline p could be found analytically after solving of the system of process and conditions equations or graphically with descriptive geometry methods (Holliday-Darr, 1998).

In the last case the 2D projections of shapes are used instead of real 3D forms (Figure 2).

The cone intersects with the cylinder. To find the intersection line, we have to apply projections of an auxiliary form, for example, a sphere. The sphere intersects the cylinder and the cone by two circles. These circles cross in two points P_1 and P_2 . On the Figure 2 the projections of these points are marked as p_1 and p_2 on plane H and as p'_1 and p'_2 on plane V .

Using N auxiliary spheres of various radiuses and center points we can get projections of an enough number of such points P_1, \dots, P_N which compose the intersection line of the cone and the cylinder. It is the solvation of (3).

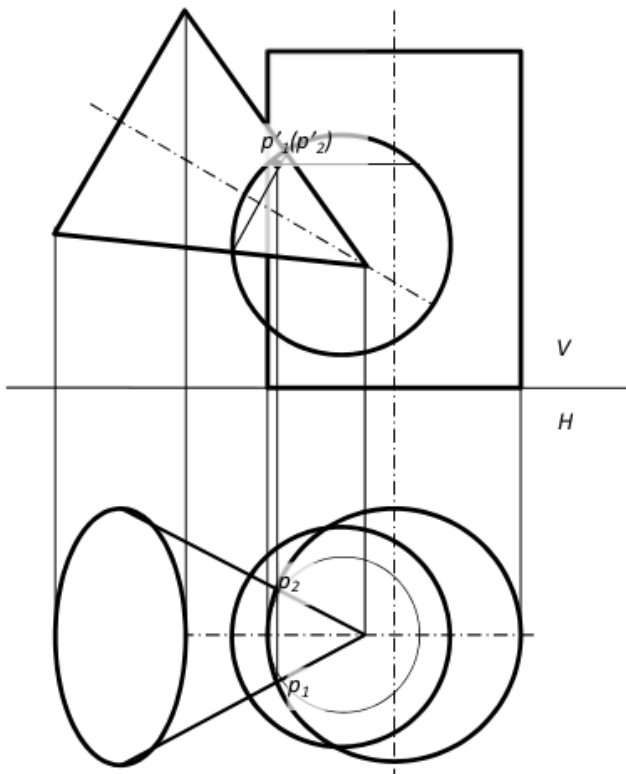


Figure 2. Intersection in projections

In the case of more than 3 dimensions, the problem could be solved by the same method, but we need more than 2 flat projections. So, this method is appropriate for every multicriterial problem.

The sphere is a very simple body where all points are on the same distance from the sphere center. This singularity allows to use a sphere as auxiliary shape for a wide range of graphical tasks. For example, we store the data of our shapes projections in the database in tables of two columns each. Every record corresponds to one projection of one point on some plane. Such discrete data format matches a bitmap computer image. Putting the auxiliary sphere center inside the cross bodies we are able to get projections of the sphere in forms of lines, circles or ellipses presented as records of discrete points. With simple database SQL operations like SELECT ... WHERE ... we can find the points of intersection of sphere and original shapes projections.

3. Materials and Methods

Geometric algebra operates shapes using algebraic operations. Sometimes it calls Clifford algebra, but geometric algebra is only a special case of Clifford algebra. The methods of geometric algebra are being used for moving pictures of 3D objects on the computer screen. Like in projective geometry the simple flat forms are used for creating complex images (Dorst et al., 2007). For instance, a circle is described by three points c_1, c_2 and c_3 .

$$C = c_1 \wedge c_2 \wedge c_3. \quad (4)$$

where \wedge symbol means the construction of a new geometric element through the points. The constructed element can be rotated with element R called a rotor

$$C \mapsto R C / R. \quad (5)$$

Notice that we are speaking about objects in 3D space. The line L can be constructed with points a_1, a_2 and infinity ∞

$$L = a_1 \wedge a_2 \wedge \infty. \quad (6)$$

Line formula (6) is very similar to circle formula (4). To rotate something around the line L , we must use the rotor R

$$R = \exp(\phi L^* / 2), \quad (7)$$

where ϕ is the rotation angle and L^* is the dual of L .

Similar to line the plane Π is described by three points p_1, p_2, p_3 and infinity ∞

$$\Pi = p_1 \wedge p_2 \wedge p_3 \wedge \infty. \quad (8)$$

Another plane π can be specified by the normal vector \mathbf{n} , the point p in π or the vector \mathbf{p} perpendicular to \mathbf{n}

$$\pi = p \downarrow (\mathbf{n}\infty) = \mathbf{n} - (\mathbf{p} \cdot \mathbf{n})\infty. \quad (9)$$

The most important operation is a reflection which can be described as

$$X \mapsto -\pi X/\pi, \tag{10}$$

where X is some reflected geometric element and π is the plane of reflection.

Software implementation of geometric algebra operators is performed in Ganja program project.

Table 2. Some shape description

Shape	Description in Ganja.js
Cylinder	<code>cylinder = (r=1,h=1,x=32)=>wrap(lathe(!1e0,!1e0+r*1e3),!(1e0+r*1e3+h*1e1),!(1e0+h*1e1)],x,1e23))</code>
Torus	<code>torus = (r=.3,r2=.25,x=32,y=16)=>wrap(lathe((1+r*.5e03)>>>lathe(!1e0+r2*(1e1+1e3)/2**5),y,1e13),x,1e23))</code>
Sphere	<code>sphere = (r=1,x=32,y=16)=>wrap(lathe(lathe(!1e0+r*1e1),y,1e13,.5),x,1e23))</code>
Cone	<code>cone = (r=1,h=1,x=64)=>wrap(lathe(!1e0,!1e0+r*1e3),!(1e0+h*1e1)],x,1e23))</code>
Arrow	<code>arrow = ()=>[...cone(.15,.3),...cone(.15,0),...cylinder(.05,-2)]</code>

The example of two shapes intersection is on Figure 3.

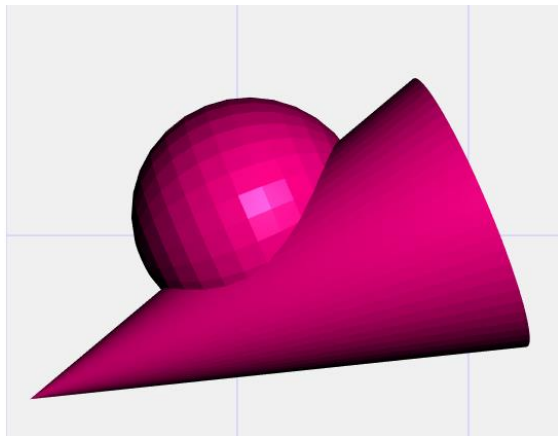


Figure 3. An intersection of two shapes approximating areas corresponding optimal solutions and boundary conditions

Notice that Ganja allows to simulate the objects intersection projection in dynamic. For instance, a multicriterial process and its border conditions can be modeled by two shapes. Time including means that shapes are moving relative to each other. So, the line of intersection is changing its form and size depending on time.

4. Results and Discussion

Geometric programming solves non-linear optimization tasks in multidimensional space. The same problems are being solved by convex optimization (Boyd and Vandenberghe, 2004). The special case of such type of problems is the classic control problem. Let a dynamic system with discrete time points $t=0,1,2,\dots,T$ controlled by impacts $u_t \in R_m$. The system function $x_t \in R_n$ is expressed as

$$x_{t+1} = Ax_t + Bu_t, \tag{11}$$

where $A \in R^{n*n}$ and $B \in R^{n*m}$ are determined matrixes. We have to find optimal impacts u in conditions u_0, \dots, u_{T-1}

$$\min \sum_{t=0}^{T-1} l(x_t, u_t) + l_T(x_T), \tag{12}$$

Ganja.js package is intended to solve geometric algebra problems in up to 32 dimensions space. The package uses JavaScript language and some symbols and denotations of geometric algebra operators. There are some examples of shapes descriptions in Table 2.

$$\text{sub} \begin{cases} x_{t+1} = Ax_t + Bu_t \\ (x_t, u_t) \in C, x_T \in C_T \end{cases}$$

where $l: R_n \times R_m \rightarrow R$ is every step's cost, l_T is a final cost, C and C_T are confines.

This problem can be solved by Python CVXPY module. The results are able to be presented numerically or graphically (Figure 4).

Figure 4 shows four parallel projections of the control functions on planes formed by axes: ot (time axis) and oX_i/oU_i (impacts/state axes).

Figures 5, 6 and 7 show attainability sets $D(t, t_0, x(t_0))$ for the investigated dynamical system. Figures illustrate three possible situation which can appear during one land control station services two satellites receiving their signals for different moments $t_i=7, t_3=12$. Values along horizontal and vertical axes correspond common sent information volumes for different cases of information receiving and processing technology. Figure 5 demonstrates the absence of conflicts when interaction zones (IZ) do not intersect. Figure 6 shows the full intersection of IZs with maximal conflicts. The intermediate situation is on Figure 7 when IZs partially intersect each other.

The analysis of these diagrams displays the geometrical method advantages for throughput estimation.

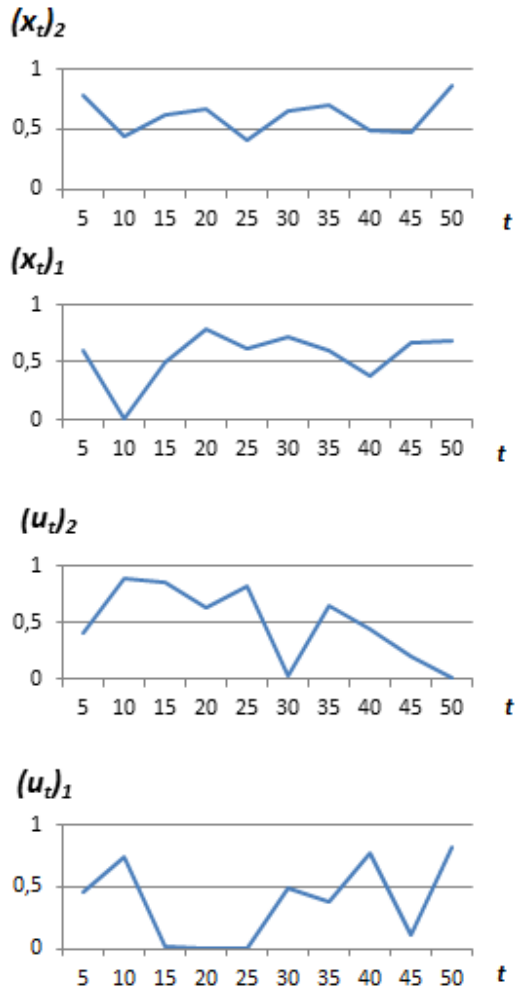


Figure 4. Diagrams might be interpreted as the solution projections onto planes uot , xot , etc.

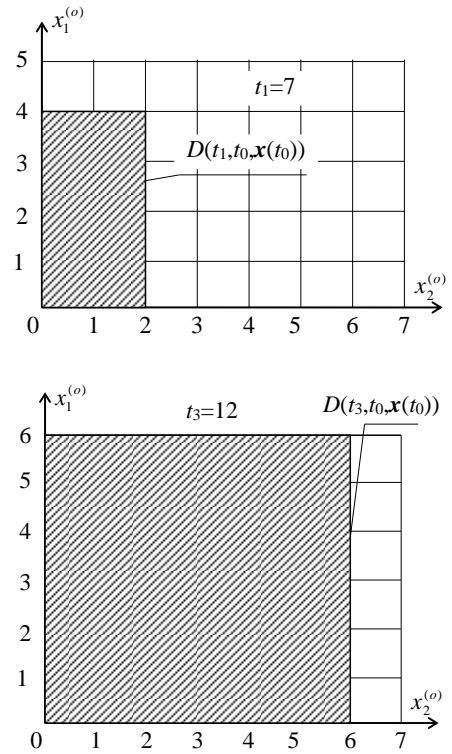


Figure 5. IZs do not intersect

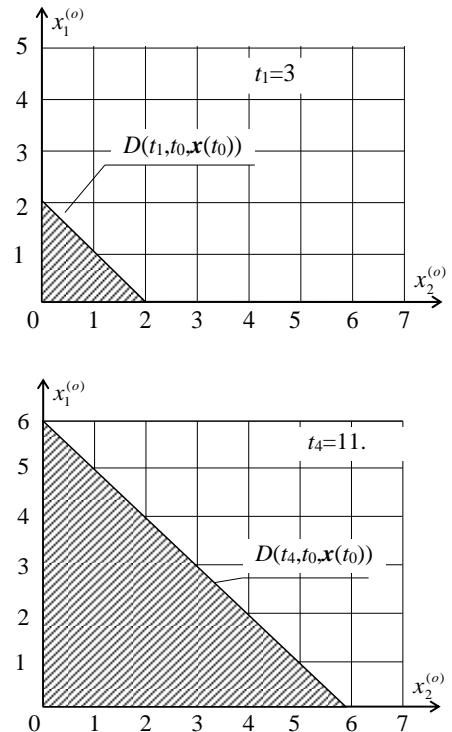


Figure 6. The full intersection of IZs

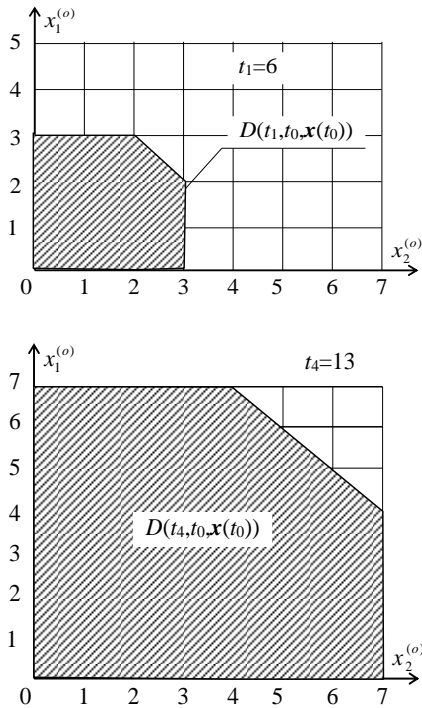


Figure 7. IZs partially intersect each other

5. Conclusions

Thus, the control function of a multicriterial system can be produced analytically or graphically. There are a lot of program applications which solve such tasks by different methods of mathematics. The geometric solutions are more appropriate by simpler calculations and better visualization of results. Some packages can be used for solving optimization problems with geometric algebra or programming geometry methods. We suggest using SQL language instructions to solve optimization problems by projective geometry methods. The best solution depends on the problem dimension, possible function approximation and available computing resources.

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