33rd European Modeling & Simulation Symposium
18th International Multidisciplinary Modeling & Simulation Multiconference

ISSN 2724-0029 ISBN 978-88-85741-57-7 © 2021 The Authors.
doi: 10.46354/i3m.2021.emss.008

Statistical simulation of the Gaussian random process parameter estimation

Oleg Chernoyarov[1,2,3,*], Larisa Korableva[4], Yury Korchagin[5], Alexander Makarov[2] and Michail Turbin[5]

[1]National Research Tomsk State University, Lenin Avenue 36, Tomsk, 634050, Russia
[2]National Research University "Moscow Power Engineering Institute", Krasnokazarmennaya st. 14, Moscow, 111250, Russia
[3]Maikop State Technological University, Pervomayskaya st. 191, Maikop, 385000, Russia
[4]Moscow State University, Leninskiye Gory 1, Moscow, 119991, Russia
[5]Voronezh State University, Universitetskaya sq. 1, Voronezh, 394018, Russia

[*]Corresponding author. Email address: chernoyarovov@mpei.ru

Abstract

One introduces the effective technique for the fast generation of the discrete samples of the logarithm of the functional of the likelihood ratio under a low-frequency Gaussian random process with the spectral density of an arbitrary shape being received against the background Gaussian white noise. The random processes that are part of the specified functional are presented by means of a finite set of discrete Fourier transform coefficients. The possible hardware and software implementation of the algorithms for estimating the frequency and energy parameters of a Gaussian process with the experimental determination of the characteristics of their performance are demonstrated. Based on the results obtained, the operation of various detectors and measurers is simulated for the case when the observable realization can be presented in the form of linear or nonlinear transformations of a Gaussian random process.

Keywords: Random process; FLR logarithm; waveform digitization; discrete Fourier transform; probability density; characteristics of estimate; statistical simulation

1. Introduction

The statistical character of many objects and phenomena studied in radio engineering requires the application of the statistical algorithms for processing the observable data so that to extract the useful information most effectively. However, generally, one can theoretically estimate the characteristics of this or that processing algorithm only in the asymptotic case provided that the unlimited signal-to-noise ratio increase takes place. For the finite values of the analyzed process parameters, the software or hardware statistical simulation can be used to define the quality of the algorithm operation. It is obvious that such simulation should lead to the minimized time cost and to the elimination of the significant errors that could arise due to the limitation of the finite size of the machine word and the inexact reconstruction of the pulse (frequency) responses of the separate processing blocks, etc. The purpose of the present paper is to introduce some software methods that can increase the efficiency of the technical implementation of the algorithms used for processing the random processes.

2. The problem statement

One starts with the study of the technique of simulation of the algorithms for processing the random processes exemplified by the algorithms for processing the fast-fluctuating low-frequency Gaussian random processes with the unknown parameters and observed against the interferences. As a mathematical model for such signals there can be used the additive combination of the form (Middleton, 1996; Van Trees, Bell, & Tian, 2013; Trifonov, Nechaev, & Parfenov, 1991):

$$x(t) = \xi(t, \vec{\vartheta}_0) + n(t), \quad t \in [0, T]. \tag{1}$$

Here $\xi(t, \vec{\vartheta}_0)$ is the realization of the centered stationary Gaussian random process with the spectral density $G(\omega, \vec{\vartheta}_0)$, $\vec{\vartheta}_0 \in \Theta$ is the vector of unknown parameters from the domain Θ, $n(t)$ is the Gaussian white noise with the one-sided spectral density N_0, $[0, T]$ is the observation interval.

The spectral density of the process $\xi(t, \vec{\vartheta}_0)$ can be presented as (Van Trees et al, 2013; Trifonov et al, 1991; Chernoyarov, Golpayegani, Zakharov, & Kalashnikov, 2020):

$$G_\xi(\omega, \vec{\vartheta}_0) = (\gamma_0/2) f(\omega/\Omega_0), \quad \vec{\vartheta}_0 = (\gamma_0, \Omega_0) \tag{2}$$

or

$$G_\xi(\omega, \vec{\vartheta}_0) = (2\pi p_0/\Omega_0) f(\omega/\Omega_0), \quad \vec{\vartheta}_0 = (p_0, \Omega_0). \tag{3}$$

In (2), (3), the notations are the following: $\Omega_0 = \int_{-\infty}^{\infty} G_\xi(\omega, \vec{\vartheta}_0)\, d\omega / \max G_\xi(\omega, \vec{\vartheta}_0)$ is the effective bandwidth, γ_0 is the intensity, p_0 is the dispersion (mean power) of the process $\xi(t, \vec{\vartheta}_0)$ so that $p_0 = \gamma_0 \Omega_0 / 4\pi$, and $f(x)$ is the function that describes the form of the spectral density and satisfies the conditions

$$f(x) \geq 0, \quad f(x) = f(-x), \quad \int_{-\infty}^{\infty} f(x)\, dx = 1. \tag{4}$$

It is presupposed that the fluctuations of the process $\xi(t, \vec{\vartheta}_0)$ are "fast", that is, the duration of the observation interval T essentially exceeds the correlation time $4\pi/\Omega_0$ of the process $\xi(t, \vec{\vartheta}_0)$ and then the following relation holds

$$\mu_0 = T\Omega_0/4\pi >> 1. \tag{5}$$

The useful information is provided by the values of the unknown parameters of the random process $\xi(t, \vec{\vartheta}_0)$ (1). Therefore, by the observable realization (1), it is necessary to estimate the vector of parameters $\vec{\vartheta}_0 \in \Theta$, $\Theta \equiv [0, \infty) \times [\Omega_{\min}, \Omega_{\max}]$ (2) or (3), where $[0, \infty)$ and $[\Omega_{\min}, \Omega_{\max}]$ are the prior intervals of the possible values of the intensity (dispersion) and bandwidth of the process $\xi(t, \vec{\vartheta}_0)$, respectively.

3. The algorithm for estimating the parameters of the fast-fluctuating random process

When synthesizing the estimation algorithm, the maximum likelihood method is used. According to this method, the logarithm of the functional of the likelihood ratio (FLR) as a function of the current values of all the unknown parameters should be generated. If the inequality (5) holds, then, based on the results of the studies (Van Trees et al, 2013; Trifonov et al, 1991; Chernoyarov, Vaculik, Shirikyan, & Salnikova, 2015), for the FLR logarithm under observations (1) one gets

$$L(\vec{\vartheta}) = \frac{1}{N_0}\int_0^T y^2(t, \vec{\vartheta})\, dt - \frac{T}{4\pi}\int_{-\infty}^{\infty} \ln\left[1 + \frac{2G_\xi(\omega, \vec{\vartheta})}{N_0}\right] d\omega, \tag{6}$$

$\vec{\vartheta} \in \Theta$.

In (6), $y(t, \vec{\vartheta}) = \int_{-\infty}^{\infty} x(t') h(t - t', \vec{\vartheta})\, dt'$ is the response of the filter to the observable realization (1) and the transfer function $H(j\omega, \vec{\vartheta})$ of this filter satisfies the condition

$$\left|H(j\omega, \vec{\vartheta})\right|^2 = H^2(\omega, \vec{\vartheta}) = G_\xi(\omega, \vec{\vartheta}) \big/ \left[G_\xi(\omega, \vec{\vartheta}) + N_0/2\right]. \tag{7}$$

Thus, the maximum likelihood estimate (MLE) of the parameters $\vec{\vartheta}_0$ is determined as

$$\vec{\vartheta}_m = \arg\sup_{\vec{\vartheta} \in \Theta} L(\vec{\vartheta}). \tag{8}$$

The numerical study of the characteristics of the estimate (8) by the Monte Carlo method requires multiple generation of the FLR logarithm (6). As it follows from (6) and (Trifonov et al, 1991; Chernoyarov et al, 2015), the FLR logarithm is a nonstationary random process and, in addition, it is the Gaussian one, if the condition (5) is fulfilled. It should be noted that the first term included in the FLR logarithm

$$L_x(\vec{\vartheta}) = \frac{1}{N_0}\int_0^T y^2(t, \vec{\vartheta})\, dt \tag{9}$$

depends on the received data (1), while the second term, which is

$$L_c(\vec{\vartheta}) = -\frac{T}{4\pi}\int_{-\infty}^{\infty} \ln\left[1 + \frac{2G_\xi(\omega, \vec{\vartheta})}{N_0}\right] d\omega, \tag{10}$$

does not depend on the realization $x(t)$ (1).

Therefore, the FLR logarithm can be presented as the sum of the two terms

$$L(\vec{\vartheta}) = L_x(\vec{\vartheta}) + L_c(\vec{\vartheta}), \tag{11}$$

where the first term $L_x(\vec{\vartheta})$ is the random process and is generated using numerical simulation and the second term $L_c(\vec{\vartheta})$ is a deterministic function that must be added to $L_x(\vec{\vartheta})$.

4. The transition to the discrete model

For the computer simulation of the algorithm (8), the analog operations included in (9), (10) should be changed by the corresponding discrete ones. As it follows from (9), to simulate the functional $L_x(\vec{\vartheta})$, it is sufficient to generate the discrete realization of the process $y(t, \vec{\vartheta})$, which is the output signal of the linear system with the amplitude-frequency characteristic (AFC) (7) responding to the realization (1). It can be seen from (7) that the AFC of the filter $H(\omega, \vec{\vartheta})$ behaves as $2G_\xi(\omega, \vec{\vartheta})/N_0$ under $\omega \to \infty$, since, by definition (2), (3), the spectral density of the random process $\xi(t, \vec{\vartheta}_0)$ is concentrated near zero frequency and occupies a limited frequency band. Hence, the cutoff frequency ω_c can be specified which satisfies the condition

$$H(\omega_c, \vec{\vartheta}) << 1 \text{ or } \omega_c >> \Omega_0/2, \tag{12}$$

and the AFC of the filter above this frequency can be set to zero. Thus, in other words, the filter with the AFC $H(\omega, \vec{\vartheta})$ (7) can be changed by the filter whose AFC is determined as

$$\tilde{H}(\omega, \vec{\vartheta}) = \begin{cases} H(\omega, \vec{\vartheta}), & |\omega| \le \omega_c, \\ 0, & |\omega| > \omega_c. \end{cases} \tag{13}$$

The error of such the change can be estimated in the following way

$$\varepsilon_Q = \int_{-\omega_c}^{\omega_c} Q(\omega, \vec{\vartheta})\, d\omega \Big/ \int_{-\infty}^{\infty} Q(\omega, \vec{\vartheta})\, d\omega .$$

Obviously, if the first or second inequality in (12) holds, then $\varepsilon_Q << 1$.

One denotes the response of a linear system with the AFC (13) to the input realization (1) as $\tilde{y}(t, \vec{\vartheta})$. It follows from (1) that it can be represented in the form

$$\tilde{y}(t, \vec{\vartheta}) = \tilde{y}_\xi(t, \vec{\vartheta}) + \tilde{y}_n(t, \vec{\vartheta}), \tag{14}$$

where $\tilde{y}_\xi(t, \vec{\vartheta}) = \int_{-\infty}^{\infty} \xi(t', \vec{\vartheta}_0)\tilde{h}(t - t', \vec{\vartheta})\, dt'$, $\tilde{y}_n(t, \vec{\vartheta}) = \int_{-\infty}^{\infty} n(t')\tilde{h}(t - t', \vec{\vartheta})\, dt'$ are the responses of this system to the signal $\xi(t, \vec{\vartheta}_0)$ and noise $n(t)$, respectively; $\tilde{h}(t, \vec{\vartheta})$ is any pulse response that the AFC (13) provides, while the corresponding phase-frequency characteristic can be arbitrary, since it is irrelevant when simulating random processes (Devroye, 1986; Chernoyarov, Sai Si Thu Min, Salnikova, Shakhtarin, & Artemenko, 2014).

In turn, according to (Devroye, 1986; Chernoyarov et al, 2014) the random process $\xi(t, \vec{\vartheta}_0)$ can be represented as the response of a linear system with the AFC

$$K_f(\omega, \vec{\vartheta}_0) = \sqrt{2G_\xi(\omega, \vec{\vartheta}_0)/N_0} \tag{15}$$

to the Gaussian white noise $n_f(t)$ with the same one-sided spectral density as the noise $n(t)$ (1), but, the processes $n_f(t)$ and $n(t)$ should not be correlated. In addition, as the relation (13) is satisfied, when simulating the responses $\tilde{y}_\xi(t, \vec{\vartheta})$ and $\tilde{y}_n(t, \vec{\vartheta})$ (14), the bandpass Gaussian random processes $\tilde{n}_f(t)$ and $\tilde{n}(t)$ with the intensities $N_0/2$ and occupying the frequency band $[-\omega_c, \omega_c]$ can be used instead of the processes $n_f(t)$ and $n(t)$ with unlimited bandwidth, so that

$$\begin{aligned} \tilde{y}_\xi(t, \vec{\vartheta}) &= \int_{-\infty}^{\infty} \tilde{n}_f(t')\tilde{h}_f(t - t', \vec{\vartheta})\, dt', \\ \tilde{y}_n(t, \vec{\vartheta}) &= \int_{-\infty}^{\infty} \tilde{n}(t')\tilde{h}(t - t', \vec{\vartheta})\, dt'. \end{aligned} \tag{16}$$

Here the pulse response $\tilde{h}(t, \vec{\vartheta})$ is such that the corresponding magnitude of the transfer function is $H_f(\omega, \vec{\vartheta}) = K_f(\omega, \vec{\vartheta}_0)\tilde{H}(\omega, \vec{\vartheta})$, and the dispersions of the processes $\tilde{n}_f(t)$, $\tilde{n}(t)$ are the same and equal to

$$\sigma_n^2 = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \frac{N_0}{2}\, d\omega = \frac{N_0\omega_c}{2\pi}. \tag{17}$$

It is easy to see that the spectral density of the process $\tilde{y}(t, \vec{\vartheta})$ (14) as the sum of the responses of the filters with the bandwidth ω_c is also limited by the frequency ω_c. Therefore, in accordance with the Nyquist theorem (Tan & Jiang 2018), the realization of the random process $\tilde{y}(t, \vec{\vartheta})$, $t \in [0, T]$ (14) can be certainly represented by the vector of the samples

$$\tilde{y}[k] = \tilde{y}(k\Delta t, \vec{\vartheta}) \tag{18}$$

taken at times $k\Delta t$, $k = \overline{0, N-1}$, $N = [T/\Delta t]$. Here the sampling step Δt is equal to π/ω_c that corresponds to the sampling rate

$$\omega_d = 2\omega_c, \tag{19}$$

while $[\cdot]$ is an integer. Then, for the required number of the simulated samples (18) one gets $N = [T\omega_c/\pi] = [T\omega_d/2\pi]$.

It follows from (12), (19) that $\omega_d >> \Omega_0$. Then, taking into account (5), the most general constraint on the model parameters can be formulated as follows:

$1 \ll \mu \ll N/2$. It means that the correlation time of the process $\tilde{y}(t, \vec{\vartheta})$ must fit within the observation interval T a large number of times, and a large number of samples (18) must be stated within the correlation time.

The next step is to present the random process (18) in the discrete form:

$$\tilde{y}[k] = \tilde{y}_\xi[k] + \tilde{y}_n[k], \tag{20}$$

where $\tilde{y}_\xi[k] = \tilde{y}_\xi(k\Delta t, \vec{\vartheta})$, $\tilde{y}_n[k] = \tilde{y}_n(k\Delta t, \vec{\vartheta})$, $k = \overline{0, N-1}$ are the discrete models of the signals $\tilde{y}_\xi(t, \vec{\vartheta})$ and $\tilde{y}_n(t, \vec{\vartheta})$ (16).

Accounting for the discretization, the dispersion (17) of the samples of the random processes $\tilde{n}_f(t)$ and $\tilde{n}(t)$ can be presented in the form

$$\sigma_n^2 = N_0\omega_c/2\pi = N_0\omega_d/4\pi = N_0 N/2T. \tag{21}$$

Using the obtained random vector $\tilde{y}[k]$ (20) and the formula (21), the discrete representation for the term (9) of the FLR logarithm depending on the observable realization (1) is introduced as follows

$$L_x(\vec{\vartheta}) = \frac{\Delta t}{N_0}\sum_{k=0}^{N-1}\tilde{y}^2[k] = \frac{1}{2\sigma_n^2}\sum_{k=0}^{N-1}\tilde{y}^2[k] = \\ = \frac{1}{2\sigma_n^2}\sum_{k=0}^{N-1}\left(\tilde{y}_\xi[k] + \tilde{y}_n[k]\right)^2,$$

which then can be transformed into the form

$$L_x(\vec{\vartheta}) = \frac{1}{2\sigma_n^2 N}\sum_{k=0}^{N-1}\left|C_{\tilde{y}_\xi}[k] + C_{\tilde{y}_n}[k]\right|^2. \tag{22}$$

Here

$$C_{\tilde{y}_\xi}[k] = \sum_{i=0}^{N-1}\tilde{y}_\xi[i]\exp\left(-j\frac{2\pi ki}{N}\right), \\ C_{\tilde{y}_n}[k] = \sum_{i=0}^{N-1}\tilde{y}_n[i]\exp\left(-j\frac{2\pi ki}{N}\right) \tag{23}$$

are the discrete Fourier transform (DFT) coefficients of the sequences $\tilde{y}_\xi[k]$ and $\tilde{y}_n[k]$, respectively.

Considering the symmetry properties of the DFT (Tan & Jiang 2018)

$$C_{\tilde{y}_\xi}[k] + C_{\tilde{y}_n}[k] = C^*_{\tilde{y}_\xi}[N-k] + C^*_{\tilde{y}_n}[N-k].$$

the number of arithmetic operations in (22) can be reduced as follows

$$L_x(\vec{\vartheta}) = \frac{1}{2\sigma_n^2 N}\left[\left|C_{\tilde{y}_\xi}[0] + C_{\tilde{y}_n}[0]\right|^2 + \right. \\ \left. + 2\sum_{k=0}^{N/2-1}\left|C_{\tilde{y}_\xi}[k] + C_{\tilde{y}_n}[k]\right|^2 + \left|C_{\tilde{y}_\xi}[N/2] + C_{\tilde{y}_n}[N/2]\right|^2\right]. \tag{24}$$

and, thereby, the simulation implementation is simplified.

The deterministic component (10) of the FLR logarithm can be approximately represented in the following way

$$L_c(\vec{\vartheta}) \approx -\frac{T}{4\pi}\int_{-\omega_c}^{\omega_c}\ln\left[1 + \frac{2G_\xi(\omega, \vec{\vartheta})}{N_0}\right]d\omega = \\ = -\frac{\omega_c T}{4\pi}\int_{-1}^{1}\ln\left[1 + \frac{2G_\xi(\omega_c\tilde{\omega}, \vec{\vartheta})}{N_0}\right]d\tilde{\omega}, \tag{25}$$

and then calculated for the specified vector $\vec{\vartheta}$ using one of the formulas of the numerical integration (Davis & Rabinowitz, 2007). For example, if the quadrangle formula is applied for this purpose, then one gets

$$L_c(\vec{\vartheta}) \approx -\frac{\omega_c T\Delta\tilde{\omega}}{4\pi}\sum_{m=0}^{M-1}\ln\left[1 + \frac{2G_\xi(\omega_c n\Delta\tilde{\omega} - \omega_c, \vec{\vartheta})}{N_0}\right]. \tag{26}$$

where $M = [2/\Delta\tilde{\omega}]$. The error of changing the integral in (25) by the sum (26) does not exceed the value $\varepsilon_c = G'_m\,\omega_c\Delta\tilde{\omega}^2/2$,

$G'_{\xi m} = \max_{\omega\in[-\omega_c, \omega_c]} G'_\xi(\omega, \vec{\vartheta}) / \left[N_0/2 + G_\xi(\omega, \vec{\vartheta})\right]$.

5. Discrete signal simulation

To calculate the DFT coefficients $C_{\tilde{y}_n}[k]$ (23) corresponding to the random vector $\tilde{y}_n[k]$, one presents the random process $\tilde{n}(t)$ (16) generating the process $\tilde{y}_n(t, \vec{\vartheta})$ as a series in sinc functions (Tan & Jiang 2018), the coefficients of which are the samples $\tilde{n}[k] = \tilde{n}(k\Delta t)$ taken with the sampling rate ω_d (19):

$$\tilde{n}(t) = \sum_{i=-\infty}^{\infty}\tilde{n}[i]\frac{\sin(\omega_c(t - i\Delta t))}{\omega_c(t - i\Delta t)}. \tag{27}$$

The process (27) corresponds to the spectrum

$$S_{\tilde{n}}(j\omega) = \Delta t\sum_{i=-\infty}^{\infty}\tilde{n}[i]\exp(-j\omega i\Delta t)\,I(\omega/\omega_d) = \\ = D_{\tilde{n}}(j\omega)\,I(\omega/\omega_d), \tag{28}$$

where the function $D_{\tilde{n}}(j\omega) = \Delta t\sum_{i=-\infty}^{\infty}\tilde{n}[i]\exp(-j\omega i\Delta t)$ is a discrete-time Fourier transform (DTFT) of the sequence $\tilde{n}[k]$ and has the period ω_d, while $I(x) = 1$, if $|x| \le 1/2$, and $I(x) = 0$, if $|x| > 1/2$.

The spectral density $S_{\tilde{y}_n}(j\omega, \vec{\vartheta})$ of the signal $\tilde{y}_n(t)$ at the output of the filter with the AFC (13) takes the form

$$S_{\tilde{y}_n}(j\omega, \vec{\vartheta}) = \tilde{H}(\omega, \vec{\vartheta})S_{\tilde{n}}(j\omega). \tag{29}$$

accurate to a nonessential phase factor. On the other hand, similarly to (28), the spectral density $S_{\tilde{y}_n}(j\omega, \vec{\vartheta})$ can be written using the DTFT of the sequence $\tilde{y}_n[k]$:

$$S_{\tilde{y}_n}(j\omega) = \Delta t \sum_{i=-\infty}^{\infty} \tilde{y}_n[i] \exp(-j\omega i \Delta t)\, I(\omega/\omega_d) = \\ = D_{\tilde{y}_n}(j\omega)\, I(\omega/\omega_d). \quad (30)$$

By comparing the formulas (29) and (30), while the expression (28) and the DTFT periodicity are taken into account, one can conclude that the DTFTs of the sequences $\tilde{y}_n[k]$ and $\tilde{n}[k]$ are related in the same way as the spectral densities of the continuous signals $\tilde{y}_n(t)$ and $\tilde{n}(t)$, namely,

$$D_{\tilde{y}_n}(j\omega) = D_{\tilde{n}}(j\omega) \sum_{i=-\infty}^{\infty} \tilde{H}(\omega - i\omega_d, \vec{\vartheta}). \quad (31)$$

Each of the sequences $\tilde{y}_n[k]$ and $\tilde{n}[k]$ include exactly N non-zero samples. Therefore, for each of them, the DFT can be calculated. It follows from (31) that the DFT coefficients of $\tilde{y}_n[k]$ and $\tilde{n}[k]$ are related by the relation

$$C_{\tilde{y}_n}[k] = C_{\tilde{n}}[k] \begin{cases} \tilde{H}(k\omega_d/N, \vec{\vartheta}), & k \le N/2, \\ \tilde{H}(\omega_d - k\omega_d/N, \vec{\vartheta}), & N/2 < k < N. \end{cases} \quad (32)$$

Carrying out similar operations for the samples $\tilde{y}_\xi[k]$ (20) and $\tilde{n}_f[k] = \tilde{n}_f(k\Delta t)$ (16), one finds that their DFT coefficients are related as

$$C_{\tilde{y}_\xi}[k] = C_{\tilde{n}_f}[k] \times \\ \times \begin{cases} K_f(k\omega_d/N, \vec{\vartheta}_0)\tilde{H}(k\omega_d/N, \vec{\vartheta}), & k \le N/2, \\ K_f(\omega_d - k\omega_d/N, \vec{\vartheta}_0)\tilde{H}(\omega_d - k\omega_d/N, \vec{\vartheta}), \\ \qquad N/2 < k < N, \end{cases} \quad (33)$$

where $K_f(\omega, \vec{\vartheta}_0)$ is defined from (15).

Based on the formulas (32) and (33), one can present the FLR logarithm (24) in the form

$$L_x(\vec{\vartheta}) = \frac{1}{2\sigma_n^2 N}\Big[\left|C_{\tilde{n}_f}[0]K_f(0, \vec{\vartheta}_0) + C_{\tilde{n}}[0]\right|^2 \tilde{H}^2(0, \vec{\vartheta}) + \\ + \left|C_{\tilde{n}_f}[N/2]K_f(\omega_d/2, \vec{\vartheta}_0) + C_{\tilde{n}}[N/2]\right|^2 \tilde{H}^2(\omega_d/2, \vec{\vartheta}) + \\ + 2\sum_{k=1}^{N/2-1}\left|C_{\tilde{n}_f}[k]K_f(\omega_d k/N, \vec{\vartheta}_0) + C_{\tilde{n}}[k]\right|^2 \tilde{H}^2(\omega_d k/N, \vec{\vartheta})\Big]. \quad (34)$$

Thus, to form the FLR logarithm $L(\vec{\vartheta})$, $\vec{\vartheta} \in \Theta$ (6), it is necessary

1) to generate the two sequences of the independent random numbers $\tilde{n}_\xi[k]$ and $\tilde{n}[k]$ with the zero mathematical expectations and dispersions σ_n^2 (21);

2) to calculate the DFT coefficients (32), (33) and the deterministic term (26) of the FLR logarithm for each possible value $\vec{\vartheta} \in \Theta$ of the vector of the unknown estimated parameters $\vec{\vartheta}_0$;

3) by substituting the found DFT coefficients into the formula (34), to calculate the values corresponding to the term of the FLR logarithm depending on the observable realization;

4) to calculate the values of the FLR logarithm according to (11).

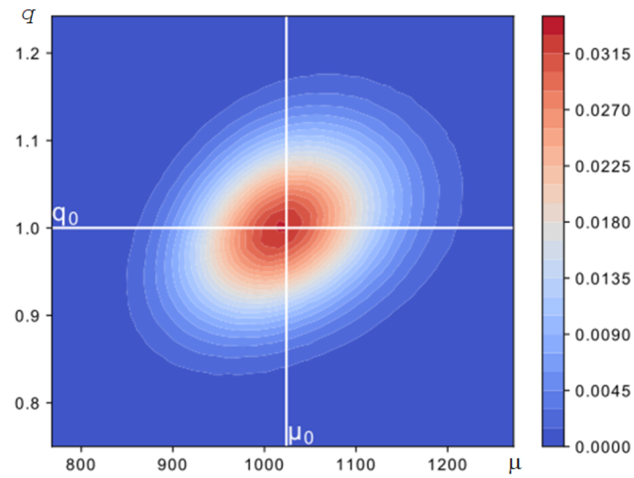
6. Simulation results

Finally, it is time to consider simulating the FLR logarithm for the Gaussian random process $\xi(t, \vec{\vartheta}_0)$ with the spectral density (3) in the case when the function (4) takes the form $f(x) = \exp(-\pi x^2)$. For convenience, following (Trifonov et al, 1991; Chernoyarov et al, 2014), one passes to the dimensionless estimated parameters. Instead of the signal $\xi(t, \vec{\vartheta}_0)$ dispersion, the value $q_0 = 4\pi p_0/N_0\Omega_0$ is estimated. It characterizes the ratio between the mean signal power and the mean noise power within the effective signal bandwidth. And instead of the effective signal bandwidth, the value (5) is estimated characterizing the ratio between the observation time and the correlation time of the random process $\xi(t, \vec{\vartheta}_0)$.

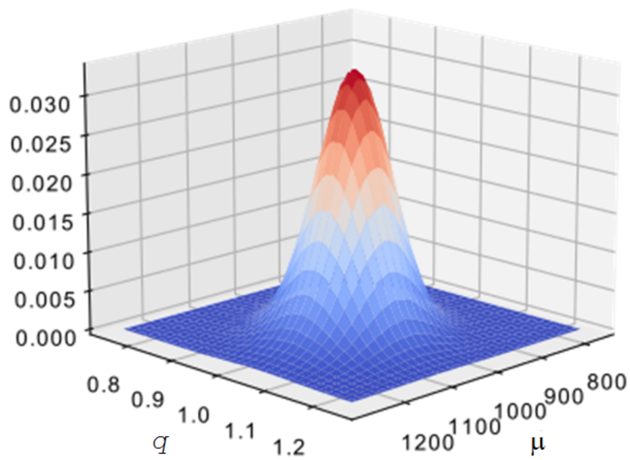
Each experiment consists of the two stages: the generation of the DFT samples of the observable realization according to the formulas (32) and (33), and then, as it follows from (8), the determination of the values of the parameters q_m and μ_m that maximize the FLR logarithm (6) generating by applying the formulas (11), (26), (34).

The simulation results, while $2.5 \cdot 10^6$ realizations (1) have been processed under the values of the parameters $q_0 = 1$ and $\mu_0 = 1024$, are shown in Figures 1 and 2. In Figures 1a and 1b, the obtained two-dimensional probability density of the position of the FLR logarithm maximum in the coordinates (q, μ) of the current values of unknown parameters is plotted. As it follows from Figure 1a, the probability density is bell-shaped (Gaussian), while it can be seen from Figure 1b that the position of the probability density maximum coincides with the real values q_0 and μ_0 of the estimated parameters.

In Figures 2, the cross-sections of the two-dimensional probability density are presented. Figures 2a and 2b demonstrate the sections of the two-dimensional probability density in the planes $\mu = \mu_0$ and $\mu = 7\mu_0/8$, respectively. Similarly, in Figures 2c and 2d, the sections of the two-dimensional probability density by the planes $q = q_0$ and $q = 7q_0/8$ are drawn.

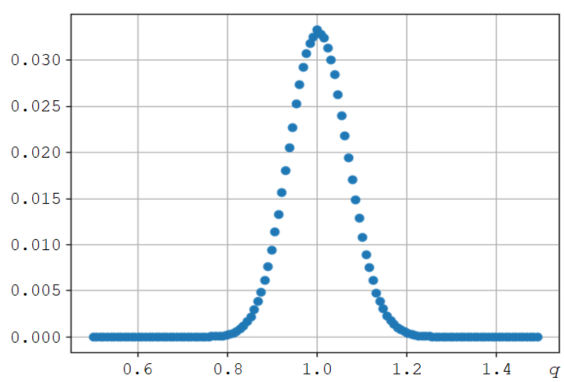


a)

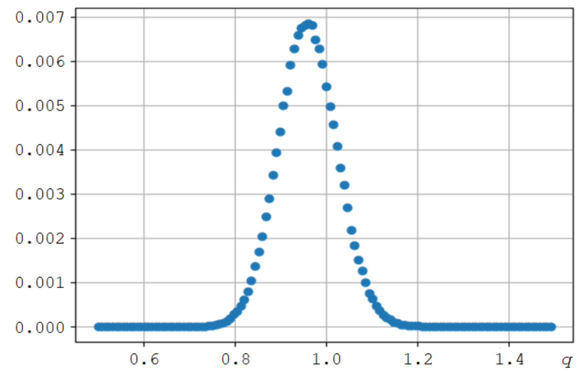


b)

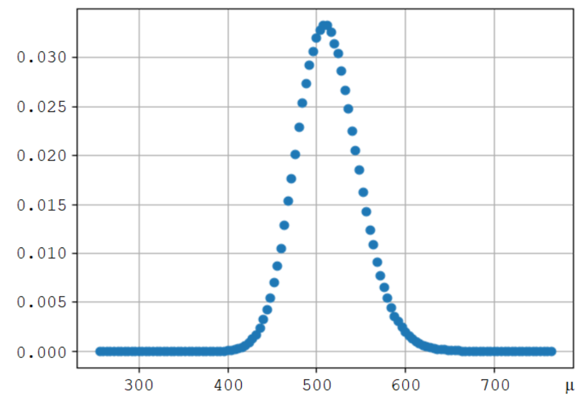
Figure 1. The two-dimensional probability density of normalized estimates of the dispersion and the bandwidth of the random process



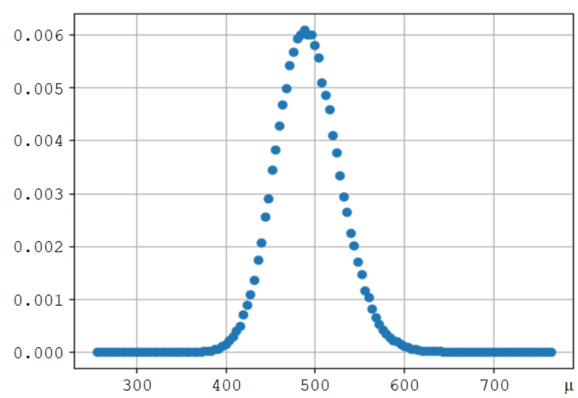
a)



b)



c)



d)

Figure 2. The cross-section of the probability density by the plane: a) $\mu=\mu_0$; b) $\mu=7\mu_0/8$; c) $q=q_0$; d) $q=7q_0/8$

It can be seen that the deviation of one parameter leads to the bias of the estimate of the other parameter.

7. Conclusions

The algorithm for simulating the estimates of the power and frequency parameters of the stationary Gaussian random process against a background of Gaussian white noise is considered. During simulation, the discrete samples of the logarithm of the functional of the likelihood ratio are multiply generated and the estimates of the dispersion and the effective bandwidth of the random process are found.

As a result, the statistical characteristics of the estimates of the specified parameters are obtained. The introduced algorithm is suitable for simulating the estimates of the parameters of a low-frequency random process with an arbitrary form of the spectral density. As an example, the characteristics of the estimates of the dispersion and the bandwidth of the Gaussian random process with the bell-shaped spectral density are studied.

The designed software for simulating the estimates is optimized for computing by a multiprocessor computer. It can work in parallel in several streams or it can exploit several computers allowing minimizing the total simulation time. The results presented can be used while determining both the biases and the variances as well as the approximate analytical expressions for the probability densities of the estimates.

Funding

This study (Section 1-5) was financially supported by the Russian Science Foundation (research project No. 20-61-47043). The simulation of the FLR logarithm for the Gaussian random process (Section 6) was carried out with support from the Ministry of Education and Science of the Russian Federation (research project No. FSWF-2020-0022).

References

Chernoyarov, O.V., Golpayegani, L.A., Zakharov, A.V., & Kalashnikov, K.S. (2020). Estimation of the stepwise random disturbance parameters with the unknown moment of appearance. *Japan Journal of Industrial and Applied Mathematics*, *37*(1), 81-102. Retrieved from https://link.springer.com/article/10.1007/s13160-019-00390-2

Chernoyarov, O.V., Sai Si Thu Min, Salnikova, A.V., Shakhtarin, B.I., & Artemenko, A.A. (2014). Application of the local Markov approximation method for the analysis of information processes processing algorithms with unknown discontinuous parameters. *Applied Mathematical Sciences, 8*(90), 4469-4496. Retrieved from http://www.m-hikari.com/ams/ams-2014/ams-89-92-2014/46415.html

Chernoyarov, O.V., Vaculik, M., Shirikyan, A., & Salnikova, A.V. (2015). Statistical analysis of fast fluctuating random signals with arbitrary-function envelope and unknown parameters. *Communications - Scientific Letters of the University of Zilina*, *17*(1A), 35-43. Retrieved from http://communications.uniza.sk/index.php/communications/article/view/410

Davis, P.J. & Rabinowitz, P. (2007). *Methods of numerical integration*. Mineola, NY: Dover Publications.

Devroye, L. (1986). *Non-uniform random variate generation*. Berlin, Germany: Springer-Verlag.

Middleton, D. (1996). *An introduction to statistical communication theory*. Hoboken, NJ: Wiley-IEEE Press.

Tan, L. & Jiang, J. (2018). *Digital signal processing*. Cambridge, MA: Academic Press.

Trifonov, A.P., Nechaev, E.P., & Parfenov, V.I. (1991). *Detection of stochastic signals with unknown parameters [in Russian]*. Voronezh, Russia: Voronezh State University.

Van Trees, H.L., Bell, K.L., & Tian, Z. (2013). *Detection, estimation, and modulation theory, Part I, Detection, estimation, and filtering theory*. New York, NY: Wiley.