33rd European Modeling & Simulation Symposium
18th International Multidisciplinary Modeling & Simulation Multiconference

ISSN 2724-0029 ISBN 978-88-85741-57-7 © 2021 The Authors.
doi: 10.46354/i3m.2021.emss.009

The characteristics of the Poisson signal source localization estimates in the regular case and in the presence of the cusp-type and change-point singularities

Oleg Chernoyarov[1,2,3,*], Alexander Faulgaber[2] and Yury Kutoyants[1,4]

[1]National Research Tomsk State University, Lenin Avenue 36, Tomsk, 634050, Russia
[2]National Research University "Moscow Power Engineering Institute", Krasnokazarmennaya st. 14, Moscow, 111250, Russia
[3]Maikop State Technological University, Pervomayskaya st. 191, Maikop, 385000, Russia
[4]Le Mans University, Avenue Olivier Messiaen, Le Mans, 72085, France

[*]Corresponding author. Email address: chernoyarovov@mpei.ru

Abstract
The problem of estimating the Poisson source localization using the inhomogeneous Poisson process observations results is considered. It is presupposed that k sensors are placed on the plane and each sensor processes the realization of the Poisson process with the intensity function depending upon the location of the source. A new mathematical theory for describing the asymptotic properties of the Bayesian and maximum likelihood estimates of the source localization is proposed. Special attention is paid to the analysis of the properties of the specified estimates depending on the regularity of the received signal front. In particular, the cases are considered when the intensity functions may be regular or may have cusp- or change-point-type singularities while their amplitudes are large. It is shown that, under the regularity conditions, the specified estimates are consistent, asymptotically normal and asymptotically efficient in terms of the minimax mean-square error. At the same time, in singular cases, only the Bayesian estimate is the effective one. Finally, some ways of implementation of the technically simple and consistent estimates of the Poisson source localization are also presented.

Keywords: Inhomogeneous Poisson process; source localization; maximum likelihood estimate; Bayesian estimate; regular parameter; cusp-type singularity; change-point singularity; statistical simulation

1. Introduction
The statistical analysis of Poisson point processes is widely used in various radio engineering applications (Chernoyarov, Kutoyants, & Zyulkov, 2019; Pchelintsev & Pergamenshchikov, 2019). Poisson processes adequately describe nuclear radiation processes (for example, in tasks of detecting radioactive materials), photoelectron flow generated by light on light-sensitive surfaces, secondary shocks subsequent to the main quake, electrical responses of the nerves to stimulation, information signals in tasks of laser ranging when detecting the objects and determining their location, in sensing and tracking tasks, etc. One of the urgent and unresolved problems of statistical analysis of Poisson processes is the

problem of localization of a radioactive source emitting a signal over the area controlled by a set of sensors. Similar situations arise when monitoring radioactive radiation, explosions, seismic activity, detecting weak optical signals, etc. (Zekavat & Buehrer, 2019).

In order to analyze Poisson flows of events, various algorithms can be used. They differ in both their characteristics and the required amount of a priori information on the signal, as well as in the complexity of the hardware implementation. Therefore, the choice of the most suitable algorithm should be based on the conditions of a specific problem, so the possibility is required to calculate and compare the characteristics of the algorithms.

Some special cases of the general problem of localizing the Poisson signal source were considered in a number of earlier studies. In particular, the least squares estimation algorithms were proposed for localizing the possibly moving source on the basis of its observations made by a fixed number of sensors (Howse, Ticknor, & Muske, 2001); there was described an iterative procedure for determining the maximum likelihood estimate (MLE) of the radioisotope source location by the measurements of its radiation; the corresponding Cramer-Rao boundary was found that defines the maximum achievable localization accuracy (Baidoo-Williams, Mudumbai, Bai, & Dasgupta, 2015); there were studied the methods for estimating the presence the source by means of the likelihood ratio and the Neumann-Pearson criterion calculations (Pahlajani, Poulakakis, & Tanner, 2013); the MLEs of the coordinates of the several emitting sources was considered (Morelande, Ristic, & Gunatilaka, 2007); the method for determining the radiation source location that applies the Bayesian approach was presented (Liu & Nehorai, 2004).

The tasks of this paper are to reveal new approaches to solving the problem of localizing a Poisson source under the presence of the singularities of various types and to study the properties of the produced estimates of the radiation source coordinates (convergence rate, limiting distribution, lower bound of the root mean square risk). The results of this study make it possible to draw conclusions on the efficiency of the presented estimation algorithms and to propose technically simple ways for their practical implementation.

The structure of the paper includes the following parts. In Section 2, the MLE and the Bayesian estimate (BE) of the Poisson signal source location on the plane are introduced. The analysis of their properties is carried out, and then it is shown that, under the regularity conditions, the specified estimates are consistent, asymptotically normal and asymptotically efficient in terms of the minimax mean-square error. In Section 3, it is presupposed that the intensity functions of the Poisson signals arriving at the sensors have a cusp-type singularity. For this very case, the consistency, the limit distributions and the convergence of the moments of the MLE and BE are evaluated. It is demonstrated that only the BE is the asymptotic efficiency estimate. In Section 4, the MLE and BE properties are considered for the case when the intensity functions of the received Poisson signals contain the points of discontinuity of the first kind. The asymptotic efficiency of the BE is indicated. The consistency, the limit distributions and the convergence of the moments of MLE and BE are described. The experimental dependences of the deviations of these estimates from the true value while the number of observations is increasing are also presented. In Section 5, the main results presented in the paper are summarized and the conclusions on the study carried out are drawn.

2. Poisson source localization on the plane: the regular case

It is presupposed that the radiation source located at the point D_0 with the coordinates $\vartheta_0 = (x_0, y_0)$ begins to radiate the signals at the moment of time $t = 0$, and the j-th sensor located at the point D_j receives data from it, which is an inhomogeneous Poisson process $X_j = (X_j(t), 0 \le t \le T)$, $j = 1 \dots k$. Such a condition, if the number of sensors is $k = 5$, is shown in Figure 1. The intensity function $\lambda_j(\vartheta_0, t) = \lambda_j(t - \tau_j) + \lambda_0$, $0 \le t \le T$ of the process X_j increases from the moment of signal appearance at the point in time $t = \tau_j$. Here $\lambda_j(t)$ is the intensity of the signal from the radiation source received by the j-th sensor, and $\lambda_j(t) = 0$, if $t \le 0$, while $\lambda_j(t) > 0$, if $t > 0$; $\lambda_0 > 0$ is the intensity of the Poisson background; $t = \tau_j$ is the time required for the signal to arrive to the j-th sensor.

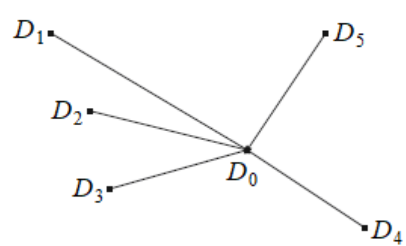

Figure 1. The localization of the signal source by a set of 5 sensors

Let $\vartheta_j = (x_j, y_j)$ are the coordinates of the j-th sensor. Then $\tau_j(\vartheta_0) = \|\vartheta_j - \vartheta_0\| / v$, where $v > 0$, is the known velocity of the signal propagation and $\|\cdot\|$ is the Euclidean norm on the plane. In order to synthesize the algorithm for estimating the position $\vartheta_0 = (x_0, y_0)$ of the radiation source by k processed realizations of independent inhomogeneous Poisson processes $X = (X_1, \dots, X_k)$, while their intensities depend upon the moments of time $\tau_j(\vartheta_0)$, the model of the observed data is represented as follows

$$\lambda_{j,n}(\vartheta_0, t) = n\lambda_j(t - \tau_j) \, 1_{\{t \ge \tau_j(\vartheta_0)\}} + n\lambda_0, \qquad (1)$$

$0 \le t \le T$.

where n is an auxiliary big parameter. Then the decision determining statistics that is the likelihood ratio $L(\vartheta, X^n)$ presented as the function of the current value ϑ of the unknown parameter ϑ_0 and the observations can take the following form:

$$\ln L(\vartheta, X^n) = \sum_{j=1}^{k} \int_{\tau_j}^{T} \ln\left[1 + \frac{\lambda_j(t - \tau_j)}{\lambda_0}\right] \mathrm{d}X_j(t) - - n\sum_{j=1}^{k} \int_{\tau_j}^{T} \lambda_j(t - \tau_j)\,\mathrm{d}t \,. \tag{2}$$

Here $\tau_j = \tau_j(\vartheta)$, while $X_j = (X_j(t), 0 \le t \le T, j = 1, \ldots, k)$ are the denumerable processes from k sensors.

When determining the value ϑ_0 by the realizations (1), the estimates obtained by means of the maximum likelihood and Bayesian approaches are used. Taking into account (2), the corresponding MLE $\hat{\vartheta}_n$ and BE $\tilde{\vartheta}_n$ can be found from the relations (Van Trees, Bell, & Tian, 2013)

$$L(\hat{\vartheta}_n, X^n) = \sup_{\vartheta\in\Theta} L(\vartheta, X^n) \,,$$
$$\tilde{\vartheta}_n = \int_\Theta \vartheta p(\vartheta) L(\vartheta, X^n)\,\mathrm{d}\vartheta \Big/ \int_\Theta p(\vartheta) L(\vartheta, X^n)\,\mathrm{d}\vartheta \,, \tag{3}$$

where $p(\vartheta)$ is a priori probability density of the random variable ϑ_0 and Θ is the domain of its possible values.

The characteristics of MLE and BE are determined by the properties of the decision determining statistics (2). It is assumed that the following conditions are satisfied:

A1) for all $j = 1, \ldots, k$ the functions $\lambda_j(t)$ are such that

$$\lambda_j(t) = 0 \text{, if } t \in [-\beta_j, 0] \text{, and } \lambda_j(t) > 0 \text{, if } t \in (0, T - \alpha_j]; \tag{4}$$

A2) the functions $\lambda_j(t)$, $j = 1, \ldots, k$ are at least twice continuously differentiable by the variable t;

A3) the Fisher information matrix is uniformly nondegenerate, that is,

$$\inf_{\vartheta\in\Theta} \inf_{|e|=1} e^{\mathrm{T}} \mathbf{I}(\vartheta)\, e > 0; \tag{5}$$

A4) there are at least three detectors that do not lie on the same line.

In (4), (5), the notations are: $\alpha_j = \inf_{\vartheta\in\Theta} \tau_j(\vartheta)$, $\beta_j = \sup_{\vartheta\in\Theta} \tau_j(\vartheta) < T$,

$$\mathbf{I}(\vartheta) = \begin{pmatrix} \langle(\mathbf{x} - \mathbf{x}_0), (\mathbf{x} - \mathbf{x}_0)\rangle_\vartheta & \langle(\mathbf{x} - \mathbf{x}_0), (\mathbf{y} - \mathbf{y}_0)\rangle_\vartheta \\ \langle(\mathbf{x} - \mathbf{x}_0), (\mathbf{y} - \mathbf{y}_0)\rangle_\vartheta & \langle(\mathbf{y} - \mathbf{y}_0), (\mathbf{y} - \mathbf{y}_0)\rangle_\vartheta \end{pmatrix},$$

$$\langle \mathbf{a}, \mathbf{b}\rangle_\vartheta = \sum_{j=1}^{k} a_j b_j J_j(\vartheta) \,, \tag{6}$$

$$J_j(\vartheta) = \frac{1}{v^2 \|\vartheta_j - \vartheta\|^2} \int_{\tau_j(\vartheta)}^{T} \frac{\lambda_j^2(t - \tau_j(\vartheta))}{\lambda_j(t - \tau_j(\vartheta)) + \lambda_0}\,\mathrm{d}t \,,$$

and $\mathbf{a}$, $\mathbf{b}$ can be any of the following vectors: $\mathbf{x} = (x_1, \ldots, x_k)$, $\mathbf{y} = (y_1, \ldots, y_k)$, $\mathbf{x}_0 = (x_{01}, \ldots, x_{0k})$.

It should be noted that, for the assumptions made, a sufficiently general model of the observed data (1) is the one taking the form

$$\lambda_{j,n}(\vartheta_0, t) = an \left| t - \tau_j(\vartheta_0) \right|^\kappa 1_{\{t \ge \tau_j(\vartheta_0)\}} + n\lambda_0 \,, \quad a > 0, \quad \kappa > 1/2 \,. \tag{7}$$

The example of the intensity (7) under $\kappa = 5/8$ is shown in Figure 2.

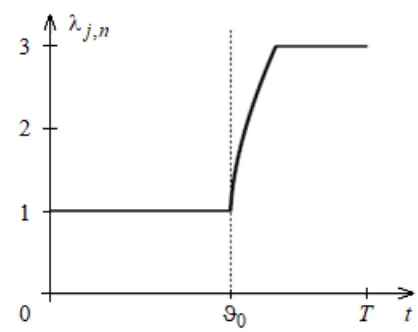

Figure 2. The intensity function of the regular Poisson process

When the conditions A1)-A4) are satisfied, then the analyzed Poisson processes and the measured parameter ϑ_0 are the regular ones. In this case, it can be shown (Chernoyarov & Kutoyants, 2020) that MLE $\hat{\vartheta}_n$ and BE $\tilde{\vartheta}_n$ are uniformly consistent, asymptotically normal, and asymptotically effective, so that

$$\sqrt{n}\left(\hat{\vartheta}_n - \vartheta_0\right) \Rightarrow N\left(0, \mathbf{I}^{-1}(\vartheta_0)\right),$$
$$\sqrt{n}\left(\tilde{\vartheta}_n - \vartheta_0\right) \Rightarrow N\left(0, \mathbf{I}^{-1}(\vartheta_0)\right). \tag{8}$$

In addition, for any $p > 0$, there is a convergence in distribution of the moments exists (Chernoyarov & Kutoyants, 2020):

$$\lim_{n\to\infty} n^{p/2} \mathbf{E}_{\vartheta_0} \left\|\hat{\vartheta}_n - \vartheta_0\right\|^p = \mathbf{E}_{\vartheta_0} \|\zeta\|^p,$$
$$\lim_{n\to\infty} n^{p/2} \mathbf{E}_{\vartheta_0} \left\|\tilde{\vartheta}_n - \vartheta_0\right\|^p = \mathbf{E}_{\vartheta_0} \|\zeta\|^p, \tag{9}$$

where $\zeta \sim N\left(0, \mathbf{I}^{-1}(\vartheta_0)\right)$.

In order to simplify the technical implementation of

MLE (3) in case when $k \geq 3$, one can apply the two-step estimation procedure, namely, at the first stage, k one-dimensional estimates $\hat{\tau}_{j,n}$ of the times of appearance τ_j, $j = 1, \dots, k$ are formed, and then the resultant estimate of the parameter ϑ_0 is constructed based on k estimates $\hat{\tau}_{j,n}$ that have been obtained.

In more details, there are k inhomogeneous Poisson processes $X^n = \left(X_1^n, \dots, X_k^n\right)$, where $X_j^n = \left(X_j(t), 0 \leq t \leq T\right)$ is a Poisson process with the intensity function

$$\lambda_{j,n}(\tau_j, t) = n\lambda_j(t - \tau_j) + n\lambda_0, \quad 0 \leq t \leq T,$$

Then the required k estimates $\hat{\tau}_{j,n}$ are determined as

$$L\left(\hat{\tau}_{j,n}, X_j^n\right) = \sup_{\tau_j \in \Theta_j} L\left(\tau_j, X_j^n\right), \quad j = 1, \dots, k,$$

where

$$L\left(\tau_j, X_j^n\right) = \exp\left[\int_{\tau_j}^{T} \ln\left[1 + \frac{\lambda_j(t - \tau_j)}{\lambda_0}\right] dX_j(t) - n\int_{\tau_j}^{T} \lambda_j(t - \tau_j) dt\right].$$

While the conditions A1)-A4) hold, MLE $\hat{\tau}_n = \left(\hat{\tau}_{1,n}, \dots, \hat{\tau}_{k,n}\right)$ is consistent, asymptotically normal: $\sqrt{n}\left(\hat{\tau}_n - \tau_0\right) \Rightarrow N\left(0, \mathbf{I}_\tau^{-1}(\vartheta_0)\right)$ and asymptotically effective. Here $\tau_0 = \left(\tau_1(\vartheta_0), \dots, \tau_k(\vartheta_0)\right)$,

$$\mathbf{I}_\tau(\vartheta_0) = \left(\mathrm{I}_{\tau j,i}(\vartheta_0)\right)_{j,i=1,\dots,k},$$

$$\mathrm{I}_{\tau j,i}(\vartheta_0) = \delta_{i,j} \int_{\tau_j(\vartheta_0)}^{T} \frac{\lambda_j'^2\left(t - \tau_j(\vartheta_0)\right)}{\lambda_j\left(t - \tau_j(\vartheta_0)\right) + \lambda_0} dt \qquad (10)$$

is the Fisher information matrix, $\delta_{i,j}$ is the Kronecker symbol.

One introduces the vector $\gamma_0 = \left(\gamma_{0,1}, \gamma_{0,2}, \gamma_{0,3}\right)$, where $\gamma_{0,1} = x_0$, $\gamma_{0,2} = y_0$, $\gamma_{0,3} = \|\vartheta_0\|^2$. It can be shown (Chernoyarov & Kutoyants, 2020) that the procedure having the form

$$\gamma_n^* = \mathbf{A}^{-1}\mathbf{Z}_n \qquad (11)$$

provides the consistent, asymptotically normal and asymptotically effective estimate γ_n^* of the vector γ_0 such that

$$\sqrt{n}\left(\gamma_n^* - \gamma_0\right) \Rightarrow N\left(0, \mathbf{D}(\vartheta_0)\right). \qquad (12)$$

In (11), (12), the notations are:

$$\mathbf{A} = \begin{pmatrix} -2\sum_{j=1}^{k} x_j & -2\sum_{j=1}^{k} y_j & k \\ -2\sum_{j=1}^{k} x_j^2 & -2\sum_{j=1}^{k} x_j y_j & \sum_{j=1}^{k} x_j \\ -2\sum_{j=1}^{k} x_j y_j & -2\sum_{j=1}^{k} y_j^2 & \sum_{j=1}^{k} y_j \end{pmatrix}, \qquad (13)$$

$$\mathbf{Z}_n = \left(\sum_{j=1}^{k}\left(z_{j,n} - r_j^2\right), \sum_{j=1}^{k} x_j\left(z_{j,n} - r_j^2\right), \sum_{j=1}^{k} y_j\left(z_{j,n} - r_j^2\right)\right),$$

$r_j^2 = x_j^2 + y_j^2$, $z_{j,n} = v^2\hat{\tau}_{j,n}^2$, $\mathbf{D}(\vartheta_0) = \mathbf{A}^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{C}\mathbf{A}^{-1}$,

$\mathbf{C} = \left(2v^2\tau_{0,j}\sigma_j, 2v^2 x_j\tau_{0,j}\sigma_j, 2v^2 y_j\tau_{0,j}\sigma_j\right)$,

$\sigma_j^2 = \mathrm{I}_{\tau j,j}^{-1}(\vartheta_0)$, and $\mathrm{I}_{\tau j,j}(\vartheta_0)$ is defined from (10).

It follows from (11), (12) that when the matrix $\mathbf{A}$ (13) is nondegenerate, then, in fulfilling the conditions A1)-A4), the estimate $\vartheta_n^* = \left(\gamma_{1,n}^*, \gamma_{2,n}^*\right)^{\mathrm{T}}$ of the parameter ϑ_0 is the consistent, asymptotically normal and asymptotically effective one, while $\sqrt{n}\left(\vartheta_n^* - \vartheta_0\right) \Rightarrow N\left(0, \mathbf{M}(\vartheta_0)\right)$, where $\mathbf{M}(\vartheta_0) = \begin{pmatrix} D_{1,1}(\vartheta_0) & D_{1,2}(\vartheta_0) \\ D_{2,1}(\vartheta_0) & D_{2,2}(\vartheta_0) \end{pmatrix}$.

It should be noted that under $\kappa = 1/2$ in (7) the decision determining statistics (2) satisfies the regularity conditions, i.e. MLE (9) is the consistent and asymptotically normal estimate, but it converges slightly faster to the true value of the estimated parameter ϑ_0 compared to the case when $\kappa > 1/2$, namely (Chernoyarov & Kutoyants, 2020): $\sqrt{n \ln n}\left(\hat{\tau}_{jn} - \tau_j(\vartheta_0)\right) \Rightarrow N\left(0, \gamma_j^{-2}\right)$, where $\gamma_j^2 = a^2/8\left(a\sqrt{T - \tau_j(\vartheta_0)} + \lambda_0\right)$.

3. Poisson source localization on the plane: the cusp case

A cusp-type singularity (Kutoyants, 1998) arises, for example, when one uses model (7) and $\kappa \in (0, 1/2)$. The example of the intensity of a Poisson signal with the cusp-type singularity for the case when $\kappa = 1/8$ is shown in Figure 3.

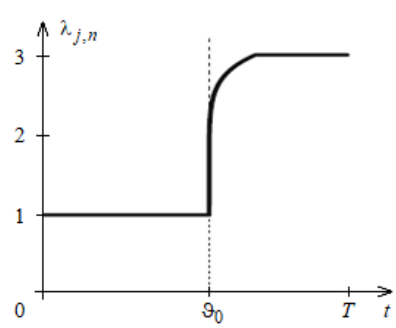

Figure 3. The intensity function of the Poisson process with the

cusp-type singularity

According to (6), for cusp-type singularities, the Fisher information matrix is not defined, and, therefore, MLE and BE (3) of the parameter ϑ_0 have different limiting distributions (Kutoyants, 1998).

It is presupposed that

B1) the source location is different from the sensor locations, i.e. there is some constant $\varepsilon > 0$ such that for each possible position of the source $\vartheta_0 \in \Theta$ the inequality $\rho_j = \left\|\vartheta_j - \vartheta_0\right\| \geq \varepsilon, \quad j = 1, \dots, k$ is satisfied;

B2) the functions $\lambda_j(s)$, $s \in \mathrm{T}$, $j = 1, \dots, k$, $\mathrm{T} = [0, T - \bar{\tau}]$, $\bar{\tau} = \max_j \sup_{\vartheta \in \Theta} \tau_j(\vartheta)$ are at least twice continuously differentiable by the variable s;

B3) there are at least three sensors that are not on the same line.

Taking into account B1)-B3) it can be shown (Chernoyarov, Dachian, & Kutoyants, 2020) that for all $\vartheta_0 \in \Theta$ and for quadratic loss function, the following inequality provides:

$$\lim_{\delta \to 0} \underline{\lim}_{n \to \infty} \inf_{\bar{\vartheta}_n} \sup_{\|\vartheta - \vartheta_0\| < \delta} n^2 \mathbf{E}_\vartheta \left\|\bar{\vartheta}_n - \vartheta\right\|^2 \geq \mathbf{E}\left\|\tilde{\zeta}\right\|^2, \tag{14}$$

Here $\tilde{\zeta} = (\tilde{\zeta}_1, \tilde{\zeta}_2)$ is the random vector $\tilde{\zeta}_i = \int_{R^2} u_i Z(u_1, u_2)\, du_1 du_2 \Big/ \int_{R^2} Z(u_1, u_2)\, du_1 du_2$, $i = 1,2$, $Z(u_1, u_2)$ is the limiting likelihood ratio, and "inf" operation is performed over all possible estimates $\bar{\vartheta}_n$ of the parameter ϑ_0. It follows from (14) that the asymptotically effective estimate ϑ_n^* is the one for which this inequality turns into equality:

$$\lim_{\delta \to 0} \underline{\lim}_{n \to \infty} \inf_{\bar{\vartheta}_n} \sup_{\|\vartheta - \vartheta_0\| < \delta} n^2 \mathbf{E}_\vartheta \left\|\vartheta_n^* - \vartheta\right\|^2 = \mathbf{E}\left\|\tilde{\zeta}\right\|^2. \tag{15}$$

In (Chernoyarov, Dachian, & Kutoyants, 2020), it is established that the relation (15) is satisfied for BE (3) only. Thus, if the decision determining statistics has cusp-type singularities, then MLE (3) is not even the asymptotically effective estimate. For the rates of convergence of MLE and BE (3), one gets

$$\begin{aligned} \mathbf{E}_{\vartheta_0}\left\|\hat{\vartheta}_n - \vartheta_0\right\|^2 &= \hat{c}\,[\,1 + o(1)\,]/n^{2/(2\kappa+1)}, \\ \mathbf{E}_{\vartheta_0}\left\|\tilde{\vartheta}_n - \vartheta_0\right\|^2 &= \tilde{c}\,[\,1 + o(1)\,]/n^{2/(2\kappa+1)}. \end{aligned} \tag{16}$$

Here $\hat{c}$, $\tilde{c}$ are some particular constants ($\hat{c} > \tilde{c}$), while $o(1)$ denotes the higher-order infinitesimal terms compared with 1. As it can be seen from a comparison of (8) and (16), in the presence of cusp-type singularities, MLE (or BE) of the Poisson signal source location has a better accuracy compared to the regular case. In addition, the smaller is the value of the parameter κ, the higher is the accuracy of estimates (3).

4. Poisson source localization on the plane: the change-point case

A change-point-type singularity arises, for example, when the observed data model is described by the expression (7) under $\kappa = 0$ as it is shown in Figure 4 (Chernoyarov & Kutoyants, 2020; Farinetto, Kutoyants, & Top, 2020).

As in the case of cusp-type singularities, the Fisher information matrix for change-point-type singularities is not defined, so that MLE and BE (3) of the parameter ϑ_0 have different limiting distributions (Kutoyants, 1998; Farinetto, Kutoyants, & Top, 2020).

One assumes that the conditions B1)-B3) specified in the Section 3 are satisfied. Then it can be shown (Chernoyarov, Dachian, & Kutoyants, 2020) that for all $\vartheta_0 \in \Theta$ and for quadratic loss function, the inequality similar to (14) holds, and it passed into the equality (15) for BE (3) only. Thus, when the decision determining statistics have change-point-type singularities, BE is the only asymptotically effective estimate. For the rates of convergence of MLE and BE (3), one gets

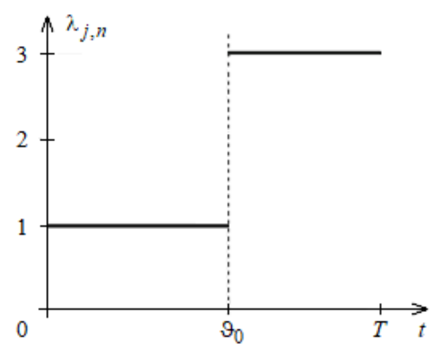

Figure 4. The intensity function of the Poisson process with the change-point-type singularity

$$\begin{aligned} \mathbf{E}_{\vartheta_0}\left\|\hat{\vartheta}_n - \vartheta_0\right\|^2 &= \hat{c}'\,[\,1 + o(1)\,]/n^2, \\ \mathbf{E}_{\vartheta_0}\left\|\tilde{\vartheta}_n - \vartheta_0\right\|^2 &= \tilde{c}'\,[\,1 + o(1)\,]/n^2, \end{aligned} \tag{17}$$

where $\hat{c}'$, $\tilde{c}'$ are some particular constants, and $\hat{c}' > \tilde{c}'$. As it can be seen from the comparison of (8), (16), (17), in the presence of change-point-type singularities MLE (or BE) of the Poisson signal source location has better accuracy compared to the case of cusp-type singularities or to the regular case.

In order to test the theoretical results obtained, statistical simulation of the maximum likelihood and Bayesian algorithms for estimating the Poisson signal source location has been carried out. While simulating, it is assumed that this source is located at the point $\vartheta_0 = (0,0)$ and is born by three sensors located at the points $\vartheta_1 = (8.5, 0)$, $\vartheta_2 = (0, 8.5)$, $\vartheta_3 = (8.5\cos(5\pi/4), 8.5\sin(5\pi/4))$ during the time interval $T = 10$. In addition, the intensity of the Poisson background is $\lambda_0 = 1$, the intensities of the signals arriving at the sensors are $\lambda_1(t) = \lambda_2(t) = \lambda_3(t) = 1$, while the velocity of signal propagation from the radiation source is $v = 1$. The parameter ϑ_0 is simulated by the random variable that is uniformly distributed in a square and described by a priori probability density $p(\vartheta) = 0.25 \cdot 1_{\{(x,y)\in[-1,1]^2\}}$.

Some results of statistical simulation are demonstrated in Figures 5, 6. In Figure 5, the evolution can be seen of the change in the Euclidean distance between the BE $\tilde{\vartheta}_n = (\tilde{x}_n, \tilde{y}_n)$ determined according to (3) and the true location of the source $\vartheta_0 = (x_0, y_0)$ while the number of observations n increases. In Figure 6, the similar dependence is presented for MLE $\hat{\vartheta}_n$ (3).

As it follows from these Figures, the distance between the points ϑ_0 and $\tilde{\vartheta}_n$ (as well as between the points ϑ_0 and $\hat{\vartheta}_n$) after the initial oscillations decreases rapidly and tends to zero with *n* increasing, that is, MLE and BE (3) are the consistent estimates. However, the BE accuracy is slightly higher than the corresponding MLE accuracy. In addition, for small values of *n*, the deviations $\|\tilde{\vartheta}_n - \vartheta_0\|$ of BE $\tilde{\vartheta}_n$ from the true value ϑ_0 are significantly less than the similar values $\|\hat{\vartheta}_n - \vartheta_0\|$.

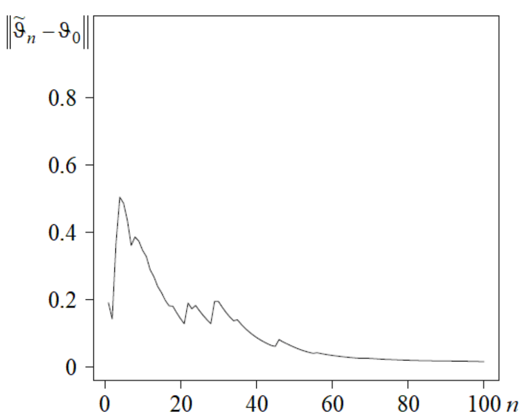

Figure 5. The dependence of the deviation of the Bayesian estimate from the true value while the number of observations is increasing

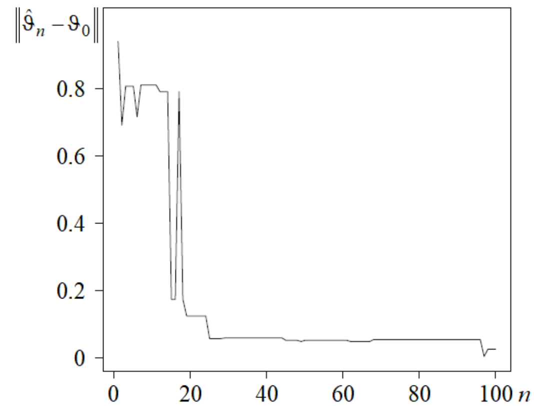

Figure 6. The dependence of the deviation of the maximum likelihood estimate from the true value while the number of observations is increasing

Thus, the Bayesian algorithm provides a better accuracy in estimating the coordinates of the source in comparison with the maximum likelihood algorithm when the amount of observations is small.

5. Conclusions

In the paper, the procedure is presented for synthesizing Bayesian and maximum likelihood algorithms for determining the location of a Poisson signal source by a set of sensors placed on a plane. It is based on the representation of the decision determining statistics as a sum of the one-dimensional random processes with its subsequent separate minimization by individual variables. By applying the generalizations of the Ibragimov-Khasminskii method based on approximating the likelihood ratio by a limiting random process independent of the observed data realization parameters, the asymptotic performance characteristics of the most commonly used algorithms for processing inhomogeneous Poisson processes with unknown regular and singular parameters are determined. It is shown that under conditions of high a posteriori accuracy, the characteristics of the maximum likelihood and Bayesian algorithms for measuring the unknown parameters of inhomogeneous Poisson signals coincide. The estimates obtained in this case are the asymptotically normal, consistent and effective ones. At the same time, in the presence of cusp- and change-point-type singularities in the measured parameters, despite both the maximum likelihood and Bayesian estimates tending to the true value of the estimated parameter while the number of observations is increasing, only the Bayesian estimate is effective. The maximum rate of convergence of the maximum likelihood and Bayesian estimates is provided in case of change-point-type singularities, while the minimum rate of convergence – in the regular case.

The results obtained allow for conclusion concerning the informed choice between the considered and the other possible algorithms for processing Poisson signals, depending on the available a priori information and the requirements for both the efficiency of the algorithm and the simplicity of its technical implementation.

Funding

This work was supported by RFBR and CNRS under Grant No. 20-51-15001.

References

Baidoo-Williams, H.E., Mudumbai, R., Bai, E., & Dasgupta, S. (2015). Some theoretical limits on nuclear source localization and tracking. *Proceedings of the 2015 Information Theory and Applications Workshop (ITA)*, pp. 270-274, February 1-6, San Diego (USA). Retrieved from https://ieeexplore.ieee.org/document/7309000

Chernoyarov, O.V., Dachian, S., & Kutoyants, Y.A. (2020). Poisson source localization on the plane: cusp case. *Annals of the Institute of Statistical Mathematics*, 72(5), 1137-1157. Retrieved from https://link.springer.com/article/10.1007%2Fs10463-019-00722-6

Chernoyarov, O.V. & Kutoyants, Y.A. (2020). Poisson source localization on the plane: the smooth case. *Metrika*, *83*(4), 411-435. Retrieved from https://link.springer.com/article/10.1007%2Fs00184-019-00738-1

Chernoyarov, O.V., Kutoyants, Y.A., & Zyulkov, A.V. (2019). On estimation errors when dealing with the problems of optical telecommunications. *Proceedings of the 31st European Modeling and Simulation Symposium (EMSS)*, pp. 90-95, September 18-20, Lisbon (Portugal). Retrieved from https://www.cal-tek.eu/proceedings/i3m/2019/emss/013/

Farinetto, C., Kutoyants, Y.A., & Top, A. (2020). Poisson source localization on the plane: change-point case. *Annals of the Institute of Statistical Mathematics*, 72(3), 675-698. Retrieved from https://link.springer.com/article/10.1007%2Fs10463-018-00704-0

Howse, J.W., Ticknor, L.O., & Muske, K.R. (2001). Least squares estimation techniques for position tracking of radioactive sources. *Automatica*, *37*(11), 1727-1737. Retrieved from https://www.sciencedirect.com/science/article/abs/pii/S0005109801001340?via%3Dihub

Kutoyants, Y.A. (1998). *Statistical inference for spatial Poisson processes. Lecture Notes in Statistics, vol. 134*. New York, NY: Springer-Verlag. Retrieved from https://link.springer.com/book/10.1007%2F978-1-4612-1706-0

Liu, Z. & Nehorai, A. (2004). Detection of particle sources with directional detector arrays. Proceedings of the 2004 Sensor Array and Multichannel Signal Processing Workshop, pp. 196-200, July 18-21, Barcelona (Spain). Retrieved from https://ieeexplore.ieee.org/document/1502936

Morelande, M.R., Ristic, B., & Gunatilaka, A. (2007). Detection and parameter estimation of multiple radioactive sources. *Proceedings of the 2007 10th International Conference on Information Fusion*, pp. 1-7, July 9-12, Quebec (Canada). Retrieved from https://ieeexplore.ieee.org/document/4408094

Pahlajani, C.D., Poulakakis, I., & Tanner, H.G. (2013). Decision making in sensor networks observing Poisson processes. *Proceedings of the 2013 21st Mediterranean Conference on Control and Automation*, pp. 1230-1235, June 25-28, Platanias-Chania (Greece). Retrieved from https://ieeexplore.ieee.org/document/6608876

Pchelintsev, E.A. & Pergamenshchikov, S.M. (2019). Model selection method for efficient signals processing from discrete data. *Proceedings of the 31st European Modeling and Simulation Symposium (EMSS)*, pp. 90-95, September 18-20, Lisbon (Portugal). Retrieved from https://www.cal-tek.eu/proceedings/i3m/2019/emss/015/

Van Trees, H.L., Bell, K.L., & Tian, Z. (2013). *Detection, estimation, and modulation theory, Part I, Detection, estimation, and filtering theory*. New York, NY: Wiley.

Zekavat, R. & Buehrer, R.M. (Eds.). (2019). *Handbook of position location: Theory, practice, and advances*. Hoboken, NJ: Wiley-IEEE Press.