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An approximation technique and a possible application for a class of delay differential equations

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Abstract

This paper deals with the analysis of an approximation method based on Differential Quadrature (DQ) rules, that represent a well-known approach to solve numerically ordinary and partial differential equations. An explicit form of the approximate solution through DQ rules is here discussed. Such a form aims to overcome some shortcomings of the traditional DQ method for a class of delay differential equations, such as the modification of the partition in order to consider the delay. Several numerical examples are presented to show the effectiveness of the approach. The analysis is completed via the presentation of a possible application for car traffic modelled by delay differential equations.

Keywords: Differential Quadrature; Pantograph; Explicit approximation; Traffic networks.

1. Introduction

There are many examples of Delay Differential Equations (DDEs) in several fields. Most popular applications are in bioscience, regarding population dynamics, epidemiology, physiology, immunology, neural networks and cell kinetics (1; 2). Recently, DDEs have found application in the agri-food field, e.g. to model pests and related dynamics (3; 4) or to simulate the effect of rainfall on some crops (5). Another interesting application of DDEs deals with road traffic modelling: in (6) the authors consider a simplified two-phase model for flows on roads, divided into road sections of homogeneous capacity and nodes for their connections. Traffic phenomena along the sections follow the Lighthill-Whitham-Richards model (LWR, see (7) and (8)), but with a simplified representation that reduces the Partial Differential Equation (PDE) approach to a DDE-based one.

In this note, the following problem is considered

$$y'(z) = r(z)y(z) + p(z)f(y(\alpha z)) + q(z), \qquad z \in [0, L], \quad (1)$$

for any $L \in R_+$, under the condition $y(z_1) = y_1$, with $y_1 \in R$, and where $r(z)$, $p(z)$, $q(z)$ are continuous functions and $0 < \alpha < 1$.

Eq. (1) is the so-called pantograph equation (9), which is a particular case of DDE with the delay as a continuous function. Many approaches for solving pantograph-type delay differential equation have been proposed (e.g. (10)-(18)). Some methods are based on a Taylor series expansion (10)-(11), but in some cases they do not provide very accurate and stable results. Other numerical methods seem to need a very dense partition in order to achieve a higher accuracy (e.g. see (13), (14), (18)). Finally, there are approaches which seem to be accurate enough, but they do not reproduce exactly the initial conditions (12) or they need a suitable initial guess for the convergence (15).

The approach herein proposed is based on Differential

Quadrature (DQ) rules. DQ rules were introduced by Bellmann and Casti in 70s (19) as an efficient way to approximate derivatives by means of a weighted sum of functional values at certain grid points. Since then, a huge literature on the application of DQ to several problems in different fields appeared (one may refer to the review articles (20), (21)). The DQ Method (DQM) is substantially a polynomial approximation and this fact can be exploited to write the approximate solution in explicit form as shown in (22). This kind of approximate solution is herein used to solve the considered class of DDEs. It should be pointed out that the usual application of DQ rules to Eq. (1) would be cumbersome, because one should consider the single delay by means of additional grid points and convergence might not be achieved (23). A numerical comparison against the solutions available in literature is presented, showing a good accuracy even when a small number of grid points is chosen.

The paper also presents a possible application for car traffic modelling, according to the approach presented in (6) and simulated in (24). Indeed, DDEs are a systematic way to foresee the behaviour of cars on roads, together with other consolidated studies, see for instance (25), (26), and (27) for real scenarios. In particular, in our case each road has two types of regimes, free and congested, whose evolution is determined by possible delays. This is exactly what normal traffic usually considers, in terms of congestion due to lack of capacities, traffic jams and spill-over effects. The regulation of flows at intersections is obtained by permeability parameters that, assumed as controls, allow discriminating the amount of traffic in the various parts of a car traffic network.

In particular, besides dynamics on roads described by DDEs, flows at intersections are found via the following rules (28): (a) the incoming traffic goes to the outgoing roads according to fixed statistical parameters; (b) drivers behave in order to maximize the flux through the intersections.

As the possible analysis of the previous car traffic model leads to "nested" equations, that cannot be solved analytically, this paper presents a simulation scenario obtained by the discretization of DDEs via DQ rules. From one side, the results are quite similar to those presented in (24). From the other, there are advantages in terms of approximation errors, as foreseen by the classical study presented for DDEs inside the paper.

The work is structured as follows. In Section 2 the methodology is presented. Section 3 is devoted to numerical experiments. Section 4 provides details for applications of DDEs to car traffic modelling. Finally, Section 5 gives some conclusions.

2. Methodology

Let $y(z)$ be a continuous function in the interval $I = [0, L]$ and let $0 = z_1 < z_2 < \ldots < z_N = L$ be a fixed, though arbitrary, partition of I with norm $h = \max_j |z_{j+1} - z_j|$. For a uniform partition, it is $h = L/(N-1)$ and $z_j = (j-1)h$. The partition can be uniform or not. In the latter case, an usual choice is the Gauss-Chebyshev-Lobatto (GCL) distribution

$$z_i = \frac{L}{2}\left[1 - \cos\frac{i-1}{N-1}\pi\right], \qquad i = 1, 2, \ldots, N. \tag{2}$$

According to the DQ rules, the pth order derivative of $y(z)$ is approximated by a weighted sum of the functional values at the grid points $y_j = y(z_j)$

$$\frac{d^p}{dz^p}y(z_i) = \sum_{j=1}^{N} A_{ij}^{(p)} y_j, \quad i = 1, \ldots, N, \tag{3}$$

where $A_{ij}^{(p)}$ are the weighting coefficients, which are usually computed as the pth−order derivative of the jth Lagrange basis polynomial at the point z_i. The explicit DQ approximate formula is (22)

$$\bar{y}(z) = y_1 + \sum_{r=1}^{N-1}\sum_{s=1}^{N} A_{1s}^{(r)} y_s \frac{z^r}{r!}. \tag{4}$$

By substituting in the given equation (1) the approximate formula (4), the following system of algebraic equations is obtained

$$\sum_{j=1}^{N} A_{ij}^{(1)} y_j = r(z_i)y_i + p(z_i)f(y_1 + \sum_{r=1}^{N-1}\sum_{s=1}^{N} A_{1s}^{(r)} y_s \frac{\alpha z_i^r}{r!}) + g(z_i), \tag{5}$$

$i = 1, \ldots, N$,, which can be easily solved with the help of computing tools.

It is worth mentioning that for initial value problems, the approximate formula (4) reproduces the initial conditions $y^{(r)}(0) = y_1^{(r)}$, with $r = 1, \ldots, p-1$ and $y_1^{(r)} \in R$, since $y^{(r)}(0) = \sum_{k=1}^{N} A_{1k}^{(r)} y_k$.

3. Numerical results

This section is devoted to numerical experiments. In all the computations, GCL points were used. The results by the proposed approach, here indicated by the acronym TDQ (standing for expliciT DQ), were compared against the solutions available in literature, in particular the ones by Taylor Series based Method (TSM) (10),(11). In all the tables, $\overline{N}$ will denote the number of terms in TSM and TDQ approaches. It should be pointed out that, for the TDQ, the number of grid points N have to be intended as $N-1$ terms in the explicit formula (4), that is $\overline{N} = N - 1$. With regard to TSM, the value of $\overline{N}$ has been increased by 1 with respect to

Table 1. Example 1: absolute errors

z	TSM ($\overline{N}$ = 9)(10)	TSM ($\overline{N}$ = 13) (10)	TDQ ($\overline{N}$ = 6)	TDQ ($\overline{N}$ = 9)	TDQ ($\overline{N}$ = 13)
0.2	1.440E - 12	2.220E - 16	1.480E-07	6.275E-12	9.812E-17
0.4	7.524E-10	1.332E-15	3.367E-07	3.408E-12	3.851E-16
0.6	2.953E-08	2.189 E - 13	4.119E-07	5.730E-12	8.525E-16
0.8	4.018E-07	9.361E-12	4.015E-07	1.516E-11	3.673E-14
1	3.059E-06	1.729E-10	5.556E-07	9.490E-11	1.151E-14

Table 2. Example 2: absolute errors, χ = 0.2

z	TSM ($\overline{N}$ = 13)(11)	CM (N=81) (13)	JRC (N=64) (14)	TDQ ($\overline{N}$ = 10)	TDQ ($\overline{N}$ = 12)
2^{-1}	1.24E-10	3.33E-11	3.93E-14	1.07E-12	1.57E-12
2^{-2}	9.74E-11	4.13E-11	2.16E-14	4.92E-14	5.14E⁻13
2^{-3}	7.00E-11	4.62E-11	1.66E-15	1.09E-14	3.07E-14
2^{-4}	9.14E-11	4.89E-11	6.21E-15	1.86E-14	1.55E-15
2^{-5}	5.28E-11	5.05E-11	3.68E-14	5.63E-14	1.33E-15
2^{-6}	1.95E-11	4.74E-11	4.54E-14	5.39E-14	1.44E-15

the one indicated in (10),(11), since the terms of the Taylor expansion in those papers were numbered starting with 0.

3.1. Example 1

This example was taken from (10). Here $r(z) = 1/2$, $p(z) = 1/2\exp(z/2)$, $f(y) = y$, $\alpha = 1/2$, $q(z) = 0$, $y(0) = 1$, $L = 1$.

The exact solution is $y(z) = \exp(z)$. The absolute errors of the approximate solution by the present method (TDQ) are tabled in Table 1 and compared against the TSM (10).

As one can see, by assuming the same $\overline{N}$, the highest absolute error by TDQ is four order lesser than the one by TSM. One can also notice that a similar (though one order lesser) highest absolute error is obtained by the TDQ, but through a smaller $\overline{N}$ (e.g. see TSM, $\overline{N}$ = 9 and TDQ, $\overline{N}$ = 6).

3.2. Example 2

The second example was taken from (11) and (12). Here $r(z) = -1$, $p(z) = \chi/2$, $f(y) = y$, $\alpha = \chi$, $q(z) = -\chi\exp(-\chi z)/2$, $y(0) = 1$, with χ a real parameter.

The exact solution is $y(z) = \exp(-z)$. The absolute errors of the approximate solution by the present method for $\chi = 0.2$ are tabled in Table 2 and compared against the TSM (11) and the most recent Continuous Method (CM) (13) and the Jacobi rational—Gauss collocation method (JRG) (14).

The errors by the proposed method are lesser than the ones by the TSM and are comparable to or slightly lesser than the ones by the other considered approaches, but through a considerably smaller number of grid points.

The case $\chi = 0.5$ was discussed in (12), where Orthogonal Exponential Polynomials (OEP) were used to solve pantograph equations. In Table 3, the results by the proposed TDQ and by using the approximate solution reported in (12) are compared. It is the case to point out that the proposed approach, differently from the one in (12), reproduces exactly the initial condition. For $N = 11$, the maximum error, in the considered range, has the same order.

Table 3. Example 2: absolute errors, χ = 0.5

z	OEP (12)	TDQ (N = 6)	TDQ (N = 11)
0	1.10E-11	0	0
0.2	6.89E-11	1.73E-06	1.45E-14
0.4	9.58E-11	4.48E-07	5.99E-13
0.6	1.01E-10	1.79E-06	8.27E-12
0.8	9.53E-11	1.11E-06	5.49E-11
1	8.40E-11	8.17E-07	2.33E-10

3.3. Example 3

This example was considered in (15), (16).

Here $p(z) = -2$, $f(y) = y^2$, $\alpha = 1/2$, $q(z) = 1$, $y(0) = 0$, $L = 1$.

The exact solution is $y(z) = \sin z$.

The absolute maximum error for solving this example by means of Homotopy Asymptotic Method is $1.2E-06$ (16) and Optimal Homotopy Asymptotic Method (OHAM) is $4E-08$ (15). It should be pointed out that the latter is referred to a polynomial of order 7. Besides, OHAM needs a suitable initial guess for the convergence. By means of the proposed method with $N = 8$, that is a polynomial of order 7, the maximum absolute error is $2.58E-09$.

3.4. Example 4

The fourth example was taken from (10), (17). For this example, it is $r(z) = p(z) = -1$, $f(y) = y$, $\alpha = 0.8$, $q(z) = 0$, $y(0) = 1$.

There is no exact solution. Numerical solutions are given in Table 4. The proposed approach shows again a faster convergence with respect to TSM.

3.5. Example 5

This example was taken from (18), where a Continuous Galerkin Method (CGM) was used. It is $r(z) = -1$, $p(z) = 0.5$, $f(y) = y$, $\alpha \in \{0.1, 0.5, 0.9\}$, $q(z) = \cos(z) + \sin(z) - 0.5\sin(\alpha z)$, $y(0) = 0$, $L = 1$.

The exact solution is $y(z) = \sin(z)$. We refer herein to the same error measure adopted in (18), that is the L_∞ norm e_∞. For the piecewise cubic CG solution in (18) , for

Table 4. Example 4: numerical solutions

z	Laguerre (n=30) (17)	TSM ($\overline{N}$ = 9) (10)	TSM ($\overline{N}$ = 12) (10)	TDQ ($\overline{N}$ = 5)	TDQ ($\overline{N}$ = 7)
0	1	1	1	1	1
0.2	0.664691	0.664691	0.664691	0.664707	0.664691
0.4	0.433561	0.433561	0.433561	0.433558	0.433561
0.6	0.276482	0.276483	0.276482	0.276499	0.276482
0.8	0.171484	0.171494	0.171484	0.17149	0.171484
1	0.102670	0.102744	0.1026705	0.102671	0.10267

Table 5. Example 5: L_∞ norm

α	h	CGM (18)	TDQ
0.1	0.222	1.48E-05	2.93E-09
0.1	0.111	1.03E-06	3.09E-11
0.5	0.222	n.a.	2.96E-09
0.5	0.125	2.69E-09	2.56E-13

$\alpha = 0.1$ and $\alpha = 0.5$, the errors are listed in Table 5. As one can see, by using the same partition norms in (18), the proposed TDQ allows a higher accuracy. For the case $\alpha = 0.9$, one has by the CGM $e_\infty = 2.69E - 09$ with $h = 0.025$. Instead, by TDQ, one gets $e_\infty = 3.06E - 09$ with $h = 0.222$, that is the same order error, but by means of a partition norm ten times about higher, implying a lower computational effort.

3.6. Example 6

As the previous one, this example was taken from (18). In this example, α is not a constant, but it is $\alpha(z) = z$. Besides, $r(z) = -1$, $p(z) = 0.5$, $f(y) = y$, $q(z) = \cos(z) - \sin(z) - 0.5\sin(\alpha z)$, $y(0) = 1$, $L = 0.5$. The exact solution is $y(z) = \cos(z)$. By means of CGM, with $h = 0.1250$, one has $e_\infty = 1.8848E - 06$. By assuming a partition with the same norm, one has $e_\infty = 3.16E - 10$.

4. Application to car traffic

This section deals with an application of DDEs to car traffic. First, a possible model, already described in (6) and (24), is shortly presented. Then, some simulations are shown.

4.1. A possible model with DDEs

Following (6) and (24), a road section j, of length L_j and with maximal possible speed of cars V_j^0, has: an arrival flow $A_j(t)$, $t \geq 0$, that represents the inflow of cars into the upstream end; a departure flow $O_j(t)$, $t \geq 0$, that is the flow of cars which leave road section j at its downstream end. Functions $A_j(t)$ and $O_j(t)$ are upper limited and their bounds are defined, see (6), assuming that, for road section j, there exists the permeability parameter $\gamma_j(t) \in [0,1]$, $t \geq 0$, that describes the amount of vehicles that go out. Notice that $\gamma_j(t)$ has the following interpretation: $\gamma_j = 0$ indicates a red or amber light; $\gamma_j = 1$ corresponds to a green light, and all cars can flow out from road section j; $0 < \gamma_j < 1$ represents a green light for a situation in which not all vehicles can go out immediately from road section j, hence indicating that queues still occur nearby the road traffic node. Notice that $0 < \gamma_j \leq 1$ also indicates situations in which there are not traffic lights at road intersections, but vehicles are free to circulate via some yielding criteria. The number of delayed vehicles for road section j, ΔN_j, has an evolution described by the generic equation:

$$\dot{\Delta N_j} = RS\left(A_j\left(t - \frac{L_j}{V_j^0}\right), O_j(t)\right), \tag{6}$$

where $RS(\cdot)$ is a function that depends on the dynamics at the intersection at which road section j is connected. Notice that:

- $RS(\cdot)$ depends on two traffic rules, see (28): (a) at each traffic nodes, drivers distribute according to fixed parameters; (b) respecting rule (a), drivers behave so as to maximize the car flux at the traffic node.
- as $\frac{L_j}{V_j^0} \geq 0$, $t - \frac{L_j}{V_j^0}$ represents a delay term that can be easily expressed as αz, by a variable change, with $0 < \alpha < 1$. This suggests that (6) is seen as a pantograph-type DDE, where the dependence on permeability parameters is expressed by the departure flows.

As (6) clearly depends on the behaviour of cars in other parts of the network, the evolutions are represented as a control system of form:

$$\dot{x} = f(x, \gamma, \gamma_\delta), \tag{7}$$

where x is the state (the number of delayed vehicles ΔN), γ is the control (the permeability parameters), and γ_δ are delayed controls. Formulation (7) allows either to define an optimal control problem for the minimization of queues on roads, see (6), or the definition of a precise theoretical background for the behaviour of traffic. In short, the variation of any control parameter γ influences the evolution of the couple (A, O) through RS; in turn the values of (A, O) influence themselves through RS and determine the continuous dynamics of ΔN, and so on. This suggests the adoption of needle variations (for an exhaustive presentation, see (6) and (24)) of permeability parameters to manage the overall features of the network. In particular, it is sufficient a unique variation of just a permeability parameter to provoke jumps in incoming and outgoing flows. Such a phenomenon has to be adequately discussed in a numerical way, thus justifying the adoption of DQ rules for DDEs of the model for car traffic.

4.2. A numerical simulation

We present the effect of a needle variation for a single permeability parameter for a node in a test network, whose topology is similar to the one presented in Figure 5 of (24). We have two different road intersections, indicated by (i, j) and $(i, j+1)$, three horizontal roads (R_{ij-1}, R_{ij} and R_{ij+1}) and four vertical roads (C_{ij}, C_{i+1j}, C_{i-1j+1}, C_{ij+1}). The other features of the network and simulation parameters are not reported completely here, as they are fully described in (24). Also in this case, we assume a needle variation for $\gamma_{R_{ij}}(t)$ as follows:

$$\gamma_{R_{ij}}(t) = \begin{cases} 0.5, & t \in [0, t_1] \cup]t_2, T], \\ 0.2 & t \in]t_1, t_2], \end{cases} \tag{8}$$

where $t_1 = 11.5$ min, $t_2 = 13$ min and $T = 25$ min is the total simulation time.

Figure 1 presents $\Delta N_{R_{ij}}(t)$ versus t due to the needle variation for $\gamma_{R_{ij}}(t)$. It is shown that, when $t \le t_0 = 2$, $\Delta N_{R_{ij}}(t)$ decreases as solutions of RS at nodes (i, j) and $(i, j+1)$ imply $\Delta N_{R_{ij}}(0) \ge RS\left(A_{R_{ij}}(t - t_0), O_{R_{ij}}(t)\right)$. At $t = t_1$, the needle variation for $\gamma_{R_{ij}}(t)$ creates a needle variation for $O_{R_{ij}}(t)$, that determines an immediate change of slope for $\Delta N_{R_{ij}}(t)$. At $t = t_2$, the needle variation for $\gamma_{R_{ij}}(t)$ vanishes, $\Delta N_{R_{ij}}(t)$ immediately changes its own slope and $O_{R_{ij}}(t)$ comes back to the nominal value imposed by RS at node (i, j). At $t_3 = 14.5$ min, $\Delta N_{R_{ij}}(t)$ reaches its maximal value, while $A_{R_{ij}}(t)$ follows the delayed $O_{R_{ij}}(t)$. At $t_4 = t_3 + t_0$, $\Delta N_{R_{ij}}(t)$ starts to decrease and becomes constant for $t \in [t_5, t_6[$, with $t_5 = t_2 + 2t_0$ and $t_6 = t_4 + t_0$. At t_6, $\Delta N_{R_{ij}}(t)$ starts to increase and grows until $t_7 = 20$ min, instant at which $\Delta N_{R_{ij}}(t)$ assumes the maximal value.

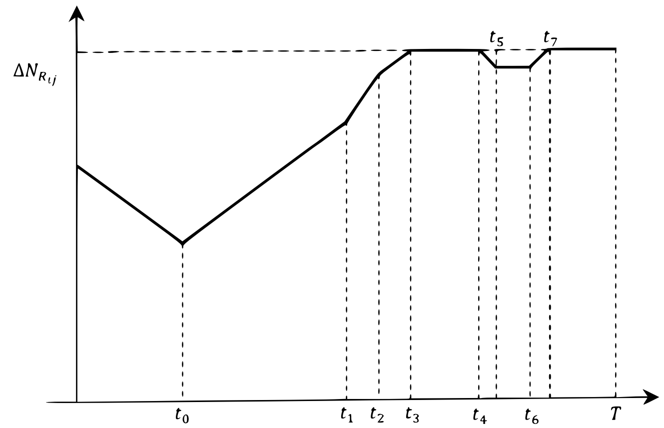

Figure 1. $\Delta N_{R_{ij}}(t)$ versus t.

5. Conclusions

In this paper, an explicit form of the approximate solution through DQ rules is used for solving a class of DDEs. The above-mentioned formula is an alternative way for applying DQ rules, aiming at providing

- the DQ solution in an explicit form, whereas the classical DQ schemes allow to find just some functional values;
- a solution procedure for a class of DDEs, without modifying the chosen partition in order to consider the delay, as implied by the usual DQ method;
- a solution which reproduces exactly the initial condition, differently from other methods used for solving DDEs, such as for instance OEP (12).

Besides, the numerical results showed that the proposed approach allows a higher accuracy, through a lesser number of grid points, when compared to some methods, such as TSM (10; 11), CM (13) , JRC (14), CGM (18).

Finally, a possible application for car traffic is presented, emphasizing the role of DDEs for road traffic models.

As a future work, a marching scheme based on this explicit form will be investigated for long-time integration, a matter not deeply discussed for the class of DDEs here considered. For car traffic models with DDEs, further studies will be developed in order to achieve optimization procedures to minimize queues on roads.

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References

1. G. A. Bocharova, F. A. Rihan (2000) Numerical modelling in biosciences using delay differential equations, J. Comp. Appl. Math. 125:183–199.

2. M. Dehghan, R. Salehi (2010) Solution of a nonlinear time-delay model in biology via semi-analytical approaches, Computer Phys. Comm. 181(7):1255-–1265,
3. F. Al Basir, A. Banerjee, S. Ray (2019) Role of farming awareness in crop pest management - A mathematical model, J. Theor. Biol. 461:59–67
4. B. Pell, A. E. Kendig, E. T. Borer, Y. Kuang (2018) Modeling nutrient and disease dynamics in a plant-pathogen system, Math. Biosci. Eng. 16(1): 234—264,
5. L.R.M. Wilson, N. C. Cryerb, E. Haughey (2019) Simulation of the effect of rainfall on farm-level cocoa yield using a delayed differential equation model, Scientia Horticulturae, 253:371–375
6. L. Rarità, C. D'Apice, B. Piccoli and D. Helbing (2010) Sensitivity analysis of permeability parameters for flows on Barcelona networks, J. Differ. Equ. 249(12): 3110 – 3131.
7. M. J. Lighthill and G. B. Whitham (1955) On kinematic waves: II. A theory of traffic on long crowded roads, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 229: 317 – 345.
8. P. I. Richards (1956) Shock waves on the highway, Oper. Res. 4: 42 – 51.
9. L. Fox, D.F. Mayers, J.A. Ockendon, A.B. Tayler (1971) On a functional differential equation, J. Inst. Math. Appl. 8: 271–307.
10. M. Sezer, A. Akyuz-Dascioglu, A Taylor method for numerical solution of generalized pantograph equations with linear functional argument (2007) J. Comp. Appl. Math. 200:217–225.
11. M. Sezer, S. Yalcinbas, M. Gulsu (2008) A Taylor polynomial approach for solving generalized pantograph equations with nonhomogenous term, Int. J. Comp. Math. 85(7):1055—1063.
12. M. M. Bahs, M. Cevik, M. Sezer (2015) Orthoexponential polynomial solutions of delay pantograph differential equations with residual error estimation, Appl. Math. Comp. 271:11—21.
13. X. Y. Li, B. Y. Wu (2014) A continuous method for nonlocal functional differential equations with delayed or advanced arguments, J. Math. Anal. Applic. 409(1):485-–493.
14. E.H. Doha, A. H. Bhrawy, D. Baleanu, R. M. Hafez (2014) A new Jacobi rational—Gauss collocation method for numerical solution of generalized pantograph equations, Appl. Num. Math. 77:43—54.
15. N. R. Anakira, A. K. Alomari, I. Hashim (2013) Optimal homotopy asymptotic method for solving delay differential equations, Math. Prob. Eng. Article ID 498902, 11 pages.
16. A. K. Alomari, M. S. Noorani, and R. Nazar (2009) Solution of delay differential equation by means of homotopy analysis method, Acta Appl. Math. 108(2):395—412.
17. C. Hwang, Y.-P. Shih (1982) Laguerre series solution of a functional differential equation, Int. J. Syst. Sci. 13(7):783—788.
18. Q. Huang, X. Xu (2016) Continuous Galerkin methods on quasi-geometric meshes for delay differential equations of pantograph type, Discr. Cont. Dyn. Sys. 36(10):5423–5443.
19. R. Bellman, J. Casti (1971) Differential quadrature and long-term integration. J. Math. Anal. Appl. 34:235–238.
20. S. Tomasiello (2011) DQ based methods: Theory and application to engineering and physical sciences, in: Handbook of Research on Computational Science and Engineering: Theory and Practice. Edited by J. Leng and W. Sharrock, IGI Global, pp. 316–346.
21. F. Tornabene, N. Fantuzzi, F. Ubertini, E. Viola (2015) Strong formulation finite element method based on differential quadrature: A survey, Appl. Mech. Rev. 67(2):50–80.
22. S. Tomasiello (2012) Some remarks on a new DQ-based method for solving a class of Volterra integro-differential equations, Appl. Math. Comp. 219:399–407.
23. W. Dong, Y. Ding, . Zhu, H. Ding (2017) Differential quadrature method for stability and sensitivity analysis of neutral delay differential systems, J. Dyn. Sys. Measur. Control, ASME Trans. 139(4):044504
24. L. Rarità (2013) A model for irregular phenomena in urban traffic, Proceedings of EMSS 2013 (25th European Modeling and Simulation Symposium), Athens, Greece, September, 25th-27th, 2013: 601–610.
25. A. Cascone, C. D'Apice, B. Piccoli, L. Rarità (2007) Optimization of traffic on road networks, Math. Models Methods Appl. Sci., 17(10):1587–1617.
26. A. Cascone, C. D'Apice, B. Piccoli, L. Rarità (2008) Circulation of car traffic in congested urban areas, Commun. Math. Sci., 6(3):765–784.
27. R. Frattaruolo, R. Manzo, L. Rarità (2008) Simulation and optimization of vehicular flows in a harbour, Proceedings of EMSS2008 (20th European Modeling and Simulation Symposium), Campora San Giovanni, Amantea (CS), Italy, September 17th-19th, 2008: 638–646.
28. M. Garavello, B. Piccoli (2006) Traffic Flow on Networks. American Institute of Mathematical Sciences, Springfield.